Matlab: Logic and Functions

In programming, it is important to be able to evaluate a statement as "true" or "false". Typically, this is done through Boolean variables. Here's an example in Matlab:

Think: False, False, True, True, True. Here is another example. Figure out what Matlab will return:

x=[5 2 3 -1 5]; x==max(x)

(See the bottom of the page for the answer¹). As you might guess, the following symbols are used for logic: ==, >=, <=, >, <. A common error is to use assignment (a single equality, =) in place of a double equality ==.

If-then statements

Now that we know how to tell if a statement is true or false, we can create what is called an "if-then" statement. For example, what will the following sequence of commands do?

```
A=[1 2;3 4];
if det(A)==0
    fprintf('The matrix is not invertible.\n')
else
    B=inv(A);
end
```

Changing A=[1 2;-2 -4];? The string shown above, The matrix is not invertible. would print to the screen. You can see that executing these lines is cumbersome- We can put them inside a *function*.

Functions in Matlab

Open the Matlab editor (type edit in the command window), and type the following function. Once you're done, save the file as myfunc01.m (it should be in the same directory as the one the command window is using).

```
function [B,c]=myfunc01(A)
% function [B,c]=myfunc01(A)
% Input: square matrix A
% Output: If A is invertible, B is the inverse and c is the determinant.
```

if det(A) == 0

 $^{1}1\ 0\ 0\ 0\ 1$

```
fprintf('The matrix is not invertible.\n');
B=[]; c=0;
else
   c=det(A);
   B=inv(A);
end
```

To run this function, we can construct any matrix A, then get the outputs. To use this function on some matrix in the command window, we would type (for example),

```
Z=[1 2;-2 -4];
[G,h]=myfunc(Z)
(in this case, Z is not invertible), or:
```

B=randn(3,3);
[H,j]=myfunc(B);

I'm using all these different variables to stress something: The variable names you use in a function file are completely separate from the variable names that are used in the command window. The variables used in the function file are local in the sense that they are erased created and erased by the function itself.

Homework:

- 1. What is the Matlab command to create the array x which holds the integers: 2, 5, 8, 11, ... 89
- 2. (Referring to the array above) What would the Matlab command be that zeros out the even-numbered indices (That is, $x(2), x(4), x(6), \dots$)?
- 3. Using double precision floating point numbers (with the rounding rule), will 1 + x > 1? Show by hand (and verify in Matlab), if (a) $x = 2^{-53}$, and (b) $x = 2^{-53} + 2^{-60}$.
- 4. Suppose that a matrix A has been defined. What does this line compute? (Hint: It has something to do with the columns of A).

R=sqrt(sum(A.*A));

5. Set up the function and script file in the next section (banditE.m and banditScript.m) and see that you can replicate the figure. The code is online. What does Matlab do when you type the following? help banditE

A Case Study in Reinforcement Learning

There are several goals for us in looking at this first case study, but mainly we want to:

- Get a first look at a mathematical model.
- Get an introduction to Matlab and some Matlab programs.

In this template model, we consider the so-called n-armed bandit. The name comes from a slang term for a slot machine, also known as a one-armed bandit.

In the n-armed bandit problem, we have n slot machines (or equivalently, one machine with n-arms), each having a different probability of winning. The overall goal is to play the slots in such a way as to maximize our winning.

The problem is that we do not know the payout value for any of the machines- Therefore, our model of this situation will consist of two parts: (1) A model of the payout for each machine, and (2) A strategy to try to maximize our winnings, even with imperfect knowledge of the payouts.

We will construct a computer model to help us implement the n-armed bandits, and to try out different strategies.

Before going much further, let's set up some notation: Let

Q(a) = The expected return for playing slot machine a

You can also think of Q(a) as the *mean* of the payoffs for slot machine a. If we knew Q(a) for each machine a, our strategy would be very simple: Play the machine with the largest payoff, and ignore the others. Of course, we do not know the exact payout for each machine, so we define:

$$Q_t(a) =$$
 Our estimation of $Q(a)$ at time t

Our general algorithm should be set up so that our estimates get better over time;

$$\lim_{t \to \infty} Q_t(a) = Q(a) \tag{1}$$

One straightforward method of proceeding is to define $Q_0(a) = 0$ for every machine a. If we play slot machine a a total of n_a times with payoffs r_1, \ldots, r_{n_a} (note that these values could be negative!), then we estimate Q(a) as the mean of these values:

$$Q_t(a) = \frac{r_1 + r_2 + \ldots + r_{n_a}}{n_a}$$

In statistical terms, we are using the sample mean to estimate the actual mean (if such a thing exists in this case).

Strategies for the *n*-armed bandit

In this section, we'll describe different strategies one might take to *learn* Q(a) while at the same time trying to optimize our winnings.

The Greedy Algorithm

This strategy is straightforward: Always play the slot machine with the largest (estimated) payoff. If a_{t+1} is the machine we'll play at time t + 1, then:

$$a_{t+1} = \arg \max \{Q_t(1), Q_t(2), \dots, Q_t(n)\}$$

where "arg" refers to the argument of the maximum (which is an integer from 1 to n corresponding to the max). If there is a tie, then choose one of them at random. For example, if

we have three machines, and $Q_1(1) = 1.5$, $Q_1(2) = -0.75$ and $Q_1(3) = 1.6$, then $a_2 = 3$ (the machine we'll play at the second iteration) will be 3).

Let's take a moment to see how we might do this in Matlab. The find command will be used to find the maximum values:

```
idx=find(x==max(x))
```

will return all indices of the vector x that are equal to the max. We'll talk about the double equality in class. You should try this command with something like:

```
x=[1 2 3 0 3];
idx=find(x==max(x));
```

and the result will be a vector, idx, will contain the values 3 and 5.

Going back to the original strategy, I think you'll see a problem- What if the estimations are wrong? Then its very possible that you'll get stuck on a suboptimal machine. This problem can be dealt with in the following way: Every once in a while, try out the other machines to see what you get. This is what we'll do in the next section.

The ϵ -Greedy Algorithm

In this algorithm we will choose, with probability ϵ , a machine totally at random. In this case, as the number of trials gets larger and larger, $n_a \to \infty$ for all machines a, and so we will be guaranteed convergence to the proper estimates of Q(a) for all a machines. On the flip side, because we're always investigating other machines every once in a while, we'll never maximize our returns.

Programming Notes

To program in the ϵ -greedy algorithm, we will be repeating a certain set of commands over and over again. That is, we will (1) Choose a machine, (2) Get a "reward". To write a program to do this, we use a "for-loop". The easiest way to see how a for-loop works is to try out some examples:

```
Temp=0;
for j=1:5
  Temp=Temp+j;
end
```

As Matlab works through these commands, initially the variable Temp is set to zero. Once we get to the for-loop, Matlab will set j = 1 and execute the commands it sees until the end line (and then it will repeat). Therefore, Matlab will do the following:

- Temp= 0, set j = 1, take 0 + 1 and assign the result back to Temp.
- Now set j = 2, take (0 + 1) + 2, and assign the result back to Temp.
- Now set j = 3, and assign Temp= 3 + 3.
- Set j = 4, and Temp= 6 + 4

- Set j = 5 and Temp= 10 + 5
- Loop is finished.

For us, we will use "pseudo-code" to give an English-like explanation of our for-loop. We will execute the loop for some set number of trials, and:

for j=1 to Number of Trials

- Select an action:
 - Sometimes choose machine 'a' at random
 - Otherwise, select machine 'a' with greatest return. Check for ties, and if there is a tie, pick on of them at random.
- Get your payoff
- Update the estimates Q

The first programming problem will be to decide on how to sometimes select a machine at random, and sometimes not. If $\epsilon = \mathbf{E}$ is the probability of this event, and N is the number of trials, then one way of selection is to set up an N-vector, greedy, that will "flag" the events-that is, on trial j, if greedy(j) = 1, choose a machine at random. Otherwise, choose using the greedy method. The following code will do just that (N is the number of trials)

```
greedy=zeros(1,N);
if E>0
  m=round(E*N); %Total number of times we should choose at random
  greedy(1:m)=ones(1,m);
  m=randperm(N); %Randomly permute the vector indices
  greedy=greedy(m);
  clear m
end
```

Let's try this out to see if it works the way we think it will. Suppose we have N = 10 trials, and E = 0.3. Then initially we have a vector of 10 zeros. The value of m is 3, so we put a "1" in the first three positions of the "greedy" vector. Lastly, randomly permute these positions.

And here's the full function. We assume that the actual rewards for each of the bandits is given in the vector Aq, and that when machine *a* is played, the sample reward will be chosen from a normal distribution with unit variance and mean Aq(a).

```
function [As,Q,R]=banditE(N,Aq,E)
%FUNCTION [As,Q,R]=banditE(N,Aq,E)
% Performs the N-armed bandit example using epsilon-greedy
%
     strategy.
%
  Inputs:
%
         N=number of trials total
%
         Aq=Actual rewards for each bandit (these are the mean rewards)
%
         E=epsilon for epsilon-greedy algorithm
%
  Outputs:
%
         As=Action selected on trial j, j=1:N
```

```
%
       Q are the reward estimates
       R is N x 1, reward at step j, j=1:N
%
numbandits=length(Aq);
                       %Number of Bandits
ActNum=zeros(numbandits,1);
                       %Keep a running sum of the number of times
                       % each action is selected.
ActVal=zeros(numbandits,1);
                       %Keep a running sum of the total reward
                       % obtained for each action.
Q=zeros(1,numbandits);
                       %Current reward estimates
As=zeros(N,1);
                       %Storage for action
R=zeros(N,1);
                       %Storage for averaging reward
\% Set up a flag so we know when to choose at random (using epsilon)
greedy=zeros(1,N);
if E>0
  m=round(E*N); %Total number of times we should choose at random
  greedy(1:m)=ones(1,m);
  m=randperm(N);
  greedy=greedy(m);
  clear m
end
if E>=1
   error('The epsilon should be between 0 and 1\n');
end
%
% Now we're ready for the main loop
for j=1:N
   %STEP ONE: SELECT AN ACTION (cQ), GET THE REWARD (cR) !
      if greedy(j)>0
         cQ=ceil(rand*numbandits);
         cR=randn+Aq(cQ);
      else
          [val,idx]=find(Q==max(Q));
         m=ceil(rand*length(idx)); %Choose a max at random
         cQ=idx(m);
         cR=randn+Aq(cQ);
      end
      R(j)=cR;
   %UPDATE FOR NEXT GO AROUND!
      As(j)=cQ;
      ActNum(cQ)=ActNum(cQ)+1;
      ActVal(cQ)=ActVal(cQ)+cR;
      Q(cQ)=ActVal(cQ)/ActNum(cQ);
```

end

Next we'll create a test bed for the routine. We will call the program 2,000 times, and each call will consist of 1,000 plays. We will set the number of bandits to 10, and change the value of ϵ from 0 to 0.01 to 0.1, and see what the average reward per play is over the 1000 plays.

Here's a script file that we'll use to call the **banditE** routine:

```
Ravg=zeros(1000,1);
E=0.1;
for j=1:2000
    m=randn(10,1);
    [As,Q,R]=banditE(1000,m,E);
    Ravg=Ravg+R;
    if mod(j,10)==0
        fprintf('On iterate %d\n',j);
    end
end
Ravg=Ravg./2000;
plot(Ravg);
```

The output of the algorithms are shown in Figure 1.



- The machine (or arm) giving the highest estimated payoff will have the highest probability.
- We will choose a machine using the probabilities. For example, if the probabilities are 0.5, 0.3, 0.2 for machines 1, 2, 3 respectively, then machine 1 would be chosen 50% of the time, machine 2 would be chosen 30% of the time, and the last machine 20% of the time.

Therefore, this algorithm will maintain an exploration of all machines so that we will not get locked onto a suboptimal machine.

Now if we have n machines with estimated payoffs recorded as:

$$Q = [Q_t(1), Q_t(2), \dots, Q_t(n)]$$

we want to construct n probabilities,

$$P = [P_t(1), P_t(2), \dots, P_t(n)]$$

The requirements for this transformation are:

- 1. $P_t(k) \ge 0$ for k = 1, 2, ... (because all probabilities are positive). Another way to say this is to say that the range of the transformation is nonnegative.
- 2. If $Q_t(a) < Q_t(b)$, then $P_t(a) < P_t(b)$. That is, the transformation must be strictly increasing for all domain values.
- 3. Finally, the sum of the probabilities must be 1.

A function that satisfies requirements 1 and 2 is the exponential function. It's range is nonnegative. It maps large negative values (large negative payoffs) to near zero probability, and it is strictly increasing. Up to this point, the transformation is:

$$\hat{P}_t(k) = \mathrm{e}^{Q_t(k)}$$

We need the probabilities to sum to 1, so we normalize the $\hat{P}_t(k)$:

$$P_t(k) = \frac{\hat{P}_t(k)}{\hat{P}_t(1) + \hat{P}_t(2) + \ldots + \hat{P}_t(n)} = \frac{\exp(Q_t(k))}{\sum_{j=1}^n \exp(Q_t(j))}$$

This is a popular technique worth remembering- We have what is called a Gibbs (or Boltzmann) distribution. We could stop at this point, but it is convenient to introduce a control parameter τ (sometimes this is referred to as the temperature of the distribution). Our final version of the transformation is given as:

$$P_t(k) = \frac{\exp\left(\frac{Q_t(k)}{\tau}\right)}{\sum_{j=1}^n \exp\left(\frac{Q_t(j)}{\tau}\right)}$$

EXERCISE: Suppose we have two probabilities, P(1) and P(2) (we left off the time index since it won't matter in this problem). Furthermore, suppose P(1) > P(2). Compute the limits of P(1) and P(2) as τ goes to zero. Compute the limits as τ goes to infinity (Hint on

this part: Use the definition, and divide numerator and denominator by $\exp(Q(1)/\tau)$ before taking the limit).

What we find from the previous exercise is that the effect of large τ (hot temperatures) makes all the probabilities about the same (so we would choose a machine almost at random). The effect of small *tau* (cold temperatures) makes the probability of choosing the best machine almost 1 (like the greedy algorithm).

In Matlab, these probabilities are easy to program. Let Q be a vector holding the current estimates of the returns, as before, and let $t = \tau$, the temperature. Then we construct a vector of probabilities using the softmax algorithm:

P=exp(Q./t);
P=P./sum(P);

Programming Comments

1. How to select action a with probability p(a)?

We could do what we did before, and create a vector of choices with those probabilities fixed, but our probabilities change. We can also use the uniform distribution, so that if x=rand, and $x \le p(1)$, use action 1. If $p(1) < x \le p(1) + p(2)$, choose action 2. If $p(1) + p(2) < x \le p(1) + p(2) + p(3)$, choose action 3, and so on. There is an easy way to do this, but it is not optimal (in terms of speed). We introduce two new Matlab functions, cumsum and histc.

The function cumsum, which means *cumulative sum*, takes a vector x as input, and outputs a vector y so that y=cumsum(x) creates:

$$y_k = \sum_{n=1}^k x_n = x_1 + x_2 + \ldots + x_k$$

For example, if x = [1, 2, 3, 4, 5], then cumsum(x) would output [1, 3, 6, 10, 15]

The function histc (for histogram count) has the form: n=histc(x,y), where the vector y is monotonically increasing. The elements of y form "bins" so that n(k) counts the number of values in x that fall between the elements y(k) (inclusive) and y(k+1) (exclusive) in the vector y. Try a particular example, like:

Bins=[0,1,2]; x=[-2, 0.25, 0.75, 1, 1.3, 2]; N=histc(x, Bins);

Bins sets up the desired intervals as [0, 1) and [1, 2) and the last value is set up as its own interval, 2. Since -2 is outside of all the intervals, it is not counted. The next two elements of x are inside the first interval, and the next two elements are inside the second interval. Thus, the output of this code fragment is N = [2, 2, 1].

Now in our particular case, we set up the bin edges (intervals) so that they are the cumulative sums. We'll then choose a number between 0 and 1 using the (uniformly) random number x = rand, and determine what interval it is in. This will be our action choice:

```
P=[0.3, 0.1, 0.2, 0.4];
BinEdges=[0, cumsum(P)];
x=rand;
Counts=histc(x,BinEdges);
ActionChoice=find(Counts==1);
```

2. We have to change our parameter τ from some initial value τ_{init} (big, so that machines are chosen almost at random) to some small final value, τ_{fin} . There are an infinite number of ways of doing this. For example, a linear change from a value a to a value b in N steps would be the equation of the line going from the point (1, a) to the point (N, b).

Exercise: Give a formula for the parameter update, τ in terms of the initial value, τ_{init} and the final value, τ_{fin} if we use a linear decrease as t ranges from 1 to N.

A more popular technique is to use the following formula, which we'll use to update many parameters. Let the initial value of the parameter be given as a, and the final value be given as b. Then the parameter p is computed as:

$$p = a \cdot \left(\frac{b}{a}\right)^{t/N} \tag{2}$$

Note that when t = 0, p = a and when t = N, $p = b^2$

"Win-Stay, Lose-Shift" Strategy

In this experiment, we interpret the strategy as: If I'm winning, make the probability of choosing that action stronger. If I'm losing, make the probability of choosing that action weaker. This brings us to the class of *pursuit* methods.

Define a^* to be the winning machine at the moment, i.e.,

$$a^* = \max_a Q_t(a)$$

The idea now is straightforward- Slightly increase the probability of choosing this winning machine, and correspondingly decrease the probability of choosing the others.

Define the probability of choosing machine a as P(a) (or, if you want to explicitly include the time index, $P_t(a)$. Then given the winning machine index as a^* , we update the current probabilities by using a parameter $\beta \in [0, 1]$:

$$P_{t+1}(a^*) = P_t(a^*) + \beta \left[1 - P_t(a^*)\right]$$

and the rest of the probabilities decrease towards zero:

$$P_{t+1}(a) = P_t(a) + \beta [0 - P_t(a)]$$

²In the C/C++ programming language, indices always start with zero, and this is leftover in this update rule. This is not a big issue, and the reader can make the appropriate change to starting with t = 1 if desired.

Exercises with the Pursuit Strategy

- 1. Suppose we have three probabilities, P_1, P_2, P_3 , and P_1 is the unique maximum. Show that, for any $\beta > 0$, the updated values still sum to 1.
- 2. Using the same values as before, show that, for any $\beta > 0$, the updated values will stay between 0 and 1- that is, If $0 \le P_i \le 1$ for all *i* before the update, then after the update, $0 \le P_i \le 1$.
- 3. Here is one way to deal with a tie (show that the updated values still sum to 1): If there are k machines with a maximum, update each via:

$$P_{t+1} = (1-\beta) * P_t + \beta/k$$

- 4. Suppose that for some fixed j, P_j is always a loser (never a max). Show that the update rule guarantees that $P_j \to 0$ as $t \to \infty$. HINT: Show that $P_j(t) = (1 \beta)^t P_j(0)$
- 5. Suppose that for some fixed j, P_j is always a winner (with no ties). Show that the update rule guarantees that $P_j \to 1$ as $t \to \infty$.

Matlab Functions softmax and winstay

Here are functions that will yield the softmax and win-stay, lose-shift strategies. Below each is a driver. Read through them carefully so that you understand what each does. We'll then ask you to put these into Matlab and comment on what you see.

```
function a=softmax(EstQ,tau)
% FUNCTION a=softmax(EstQ, tau)
%
            Estimated payoff values in EstQ (size 1 x N,
    Input:
%
            where N is the number of machines
%
            tau - "temperature": High values- the probs are all
%
            close to equal; Low values, becomes "greedy"
%
            The machine that we should play (a number between 1 and N)
   Output:
if tau==0
    fprintf('Error in the SoftMax program-\n');
    fprintf('Tau must be greater than zero\n');
    a=0;
    return
end
Temp=exp(EstQ./tau);
S1=sum(Temp);
Probs=Temp./S1; %These are the probabilities we'll use
%Select a machine using the probabilities we just computed.
x=rand;
TotalBins=histc(x,[0,cumsum(Probs)']);
a=find(TotalBins==1);
```

Here is a driver for the softmax algorithm. Note the implementation details (e.g., how the "actual" payoffs are calculated, and what the initial and final parameter values are):

%Script file to run the N-armed bandit using the softmax strategy

```
%Initializations are Here:
NumMachines=10;
ActQ=randn(NumMachines,1); %10 machines
NumPlay=1000;
                            %Play 100 times
Initialtau=10;
                            %Initial tau ("High in beginning")
Endingtau=0.5;
tau=10;
NumPlayed=zeros(NumMachines,1); %Keep a running sum of the number
                                 % of times each action is selected
ValPlayed=zeros(NumMachines,1); %Keep a running sum of the total
                                 % reward for each action
EstQ=zeros(NumMachines,1);
PayoffHistory=zeros(NumPlay,1); %Keep a record of our payoffs
for i=1:NumPlay
   %Pick a machine to play:
    a=softmax(EstQ,tau);
   %Play the machine and update EstQ, tau
   Payoff=randn+ActQ(a);
    NumPlayed(a)=NumPlayed(a)+1;
   ValPlayed(a)=ValPlayed(a)+Payoff;
    EstQ(a)=ValPlayed(a)/NumPlayed(a);
   PayoffHistory(i)=Payoff;
    tau=Initialtau*(Endingtau/Initialtau)^(i/NumPlay);
end
[v,winningmachine] = max(ActQ);
winningmachine
NumPlayed
plot(1:10,ActQ,'k',1:10,EstQ,'r')
```

Here is the function implementing the pursuit strategy (or "Win-Stay, Lose-Shift").

```
function [a, P]=winstay(EstQ,P,beta)
% function [a,P]=winstay(EstQ,P,beta)
% Input: EstQ, Estimated values of the payoffs
% P = Probabilities of playing each machine
% beta= parameter to adjust the probabilities, between 0 and 1
% Output: a = Which machine to play
% P = Probabilities for each machine
```

```
[vals,idx]=max(EstQ);
winner=idx(1); %Index of our "winning" machine
%Update the probabilities. We need to do P(winner) separately.
NumMachines=length(P);
P(winner)=P(winner)+beta*(1-P(winner));
Temp=1:NumMachines;
Temp(winner)=[]; %Temp now holds the indices of all "losers"
P(Temp)=(1-beta)*P(Temp);
%Probabilities are all updated- Choose machine a w/prob P(a)
x=rand;
TotalBins=histc(x,[0,cumsum(P)']);
a=find(TotalBins==1);
```

And its corresponding driver is below. Again, be sure to read and understand what each line of the code does:

%Script file to run the N-armed bandit using pursuit strategy

%Initializations

```
NumMachines=10;
ActQ=randn(NumMachines,1);
NumPlay=2000;
Initialbeta=0.01;
Endingbeta=0.001;
beta=Initialbeta;
NumPlayed=zeros(NumMachines,1);
ValPlayed=zeros(NumMachines,1);
EstQ=zeros(NumMachines,1);
Probs=(1/NumMachines)*ones(10,1);
for i=1:NumPlay
    %Pick a machine to play:
    [a, Probs] = winstay(EstQ, Probs, beta);
    %Play the machine and update EstQ, tau
    Payoff=randn+ActQ(a);
    NumPlayed(a)=NumPlayed(a)+1;
    ValPlayed(a)=ValPlayed(a)+Payoff;
    EstQ(a)=ValPlayed(a)/NumPlayed(a);
    beta=Initialbeta*(Endingbeta/Initialbeta)^(i/NumPlay);
end
[v,winningmachine]=max(ActQ);
winningmachine
```

NumPlayed
plot(1:10,ActQ,'k',1:10,EstQ,'r')

Homework: Implement these 4 pieces of code into Matlab, and comment on the performance of each. You might try changing the initial and final values of the parameters to see if the algorithms are *stable* to these changes. As you form your comments, recall our two competing goals for these algorithms:

- Estimate the values of the actual payoffs (more accurately, the mean payout for each machine).
- Maximize our rewards!

A Summary of Reinforcement Learning

We looked in depth at a basic problem of unsupervised learning- That of trying to find the best winning slot machine in a bank of many. This problem was unsupervised because, although we got rewards or punishments by winning or losing money, we did not know at the beginning of the problem what those payoffs would be. That is, there was no expert available to tell us if we were doing something correctly or not, we had to infer correct behavior from directly playing the machines;

We also saw that to solve this problem, we had to do a lot of *trial and error* learningthat's typical in unsupervised learning. Because an expert is not there to tell us the operating parameters, we have to explore and find them out for ourselves.

We learned some techniques for translating learning theory into mathematics, and in the process, we learned some commands in Matlab. At this stage, you should be able to read some Matlab code and interpret the output of an algorithm. Later on, we'll give you more opportunities to produce your own pieces of code.

In summary, we looked at the greedy algorithm, the ϵ -greedy algorithm, the softmax strategy, and the pursuit strategy. You might consider how closely (if at all) these algorithms would reproduce human or animal behavior if given the same task.