Homework 8 Solutions

3. Let the subspace G be the plane defined below, and consider the vector $\mathbf{x}$, where:

$$G = \left\{ \alpha_1 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \text{ such that } \alpha_1, \alpha_2 \in \mathbb{R} \right\} \qquad \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

(a) Find the projector P that takes an arbitrary vector and projects it (orthogonally) to the plane G.
SOLUTION: We can use the two columns given- That is if U is constructed using the normalized vectors (they are already orthogonal), then

$$U = \begin{bmatrix} 1/\sqrt{14} & 3/\sqrt{10} \\ 3/\sqrt{14} & -1/\sqrt{10} \\ -2/\sqrt{14} & 0 \end{bmatrix}$$

The projector is UU^T, or approximately:

$$P = \begin{bmatrix} 0.9714 & -0.0857 & -0.1429 \\ -0.0857 & 0.7429 & -0.4286 \\ -0.1429 & -0.4286 & 0.2857 \end{bmatrix}$$

(b) Find the orthogonal projection of the given $\mathbf{x}$ onto the plane G.
SOLUTION: $P\mathbf{x} \approx [0.69, -0.94, 0.42]^T$

(c) Find the distance from the plane G to the vector $\mathbf{x}$.
SOLUTION: $\|\mathbf{x} - P\mathbf{x}\| \approx 1.86$.

6. The projector onto the vector $[-1, 3]^T$ is given by

$$P = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}} = \frac{1}{10} \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$$

7. Show that $P = UU^T$ is:

- An orthogonal projector:
  - Show $P^2 = P$: $P^2 = UU^TUU^T = UIU^T = UU^T = P$
  - Show $P^T = P$: $P^T = (UU^T)^T = (U^T)^TU^T = UU^T = P$
- The range is the column space of U:

$$P\mathbf{x} = UU^T\mathbf{x} = U\left(U^T\mathbf{x}\right)$$

which is a linear combination of the columns of U.
Alternative: We showed earlier that, in terms of the columns of U:

$$UU^T\mathbf{x} = \mathbf{x}^T\mathbf{u}_1(\mathbf{u}_1) + \cdots + \mathbf{x}^T\mathbf{u}_k(\mathbf{u}_k)$$

which is an expansion of $\mathbf{x}$ in the basis given by the columns of U (which is a basis for the column space of U).