Homework 9 Solution

Compute the SVD of the matrix below by hand:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

SOLUTION: We compute the eigenvalues and eigenvectors of $A^T A$ and AA^T:

- For $A^T A$ (this computes V and Σ):

  $$A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

  The characteristic equation is therefore $(1 - \lambda)^2 - 1 = 0$ so that $\lambda = 2, 0$. Therefore, the singular values are $\sigma_1 = \sqrt{2}$, and the rest are zero. Now find eigenvectors:

  - For $\lambda = 2$, the matrix equation becomes

    $$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

    We want to normalize the vectors, so multiply it $1/\sqrt{2}$.

  - For $\lambda = 0$, the matrix equation becomes

    $$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

    We want to normalize the vectors, so multiply it $1/\sqrt{2}$.

- Continuing, we now consider AA^T:

  $$AA^T = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

  which is a diagonal matrix, so the eigenvalues are 2 and 0 (they should be the same as $A^T A$). The matrix U is the identity matrix.

Therefore, the SVD is the following (the last matrix is V^T):

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$