Homework 12: Gaussian and RBF by hand

Exercise 1: Page 186, # 2

The following exercises will consider how we might set the width of the Gaussian transfer function.

1. We will approximate:

$$\left(\int_{-b}^{b} \mathrm{e}^{-x^2}\, dx\right)^2 = \int_{-b}^{b}\int_{-b}^{b} \mathrm{e}^{-(x^2+y^2)}\, dx\, dy \approx \int\int_B \mathrm{e}^{-(x^2+y^2)}\, dB$$

   where B is the disk of radius b. Show that this last integral is:

$$\pi\left(1-\mathrm{e}^{-b^2}\right)$$

   SOLUTION: We perform a polar coordinate conversion, where $r^2 = x^2 + y^2$, and $dA = r\, dr\, d\theta$. This gives:

$$\int_0^{2\pi}\int_0^{b} \mathrm{e}^{-r^2}\, r\, dr\, d\theta = 2\pi\left(-\frac{1}{2}\mathrm{e}^{-r^2}\right|_0^b = \pi\left(1-\mathrm{e}^{-b^2}\right)$$

2. Using the previous exercise, conclude that:

$$\int_{-\infty}^{\infty} \mathrm{e}^{\frac{-x^2}{\sigma^2}}\, dx = \sigma\sqrt{\pi}$$

   SOLUTION: Using the previous exercise, we can conclude that

$$\left(\int_{-b}^{b} \mathrm{e}^{-x^2}\, dx\right)^2 = \int_0^{2\pi}\int_0^{\infty} \mathrm{e}^{-r^2}\, r\, dr\, d\theta = \pi \quad\Rightarrow\quad \int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\, dx = \sqrt{\pi}$$

   If we now let $u = x/\sigma$ so $du = dx/\sigma$, then

$$\sqrt{\pi} = \int_{-\infty}^{\infty} \mathrm{e}^{-u^2}\, du = \int_{-\infty}^{\infty} \mathrm{e}^{-x^2/\sigma^2}\, \frac{1}{\sigma}\, dx$$

   From which we get the desired result.

3. We'll make a working definition of the *width* of the Gaussian: It is the value a so that k percentage of the area is between $-a$ and a (so k is between 0 and 1). The actual value of k will be problem-dependent.

   Use the previous two exercises to show that our working definition of the "width" a, means that, given a we would like to find σ so that:

$$\int_{-a}^{a} \mathrm{e}^{\frac{-x^2}{\sigma^2}}\, dx \approx k \int_{-\infty}^{\infty} \mathrm{e}^{\frac{-x^2}{\sigma^2}}\, dx$$

   Nothing much to show here- Just make the observation.

4. Show that the last exercise implies that, if we are given k and a, then we should take σ to be:

$$\sigma = \frac{a}{\sqrt{-\ln(1-k^2)}} \tag{1}$$

SOLUTION: From our previous work,

$$\int_{-a}^{a} \mathrm{e}^{-x^2/\sigma^2}\,dx \approx \sigma\sqrt{\pi}\left(1-\mathrm{e}^{-x^2/\sigma^2}\right)$$

so that

$$\int_{-a}^{a} \mathrm{e}^{\frac{-x^2}{\sigma^2}}\,dx \approx k\int_{-\infty}^{\infty} \mathrm{e}^{\frac{-x^2}{\sigma^2}}\,dx \quad \Rightarrow \quad \sigma\sqrt{\pi}\sqrt{1-\mathrm{e}^{-x^2/\sigma^2}} = k\sigma\sqrt{\pi}$$

Solve for σ:

$$1-\mathrm{e}^{-a^2/\sigma^2} = k^2 \quad \Rightarrow \quad 1-k^2 = \mathrm{e}^{-a^2/\sigma^2} \quad \Rightarrow \quad \ln(1-k^2) = -a^2/\sigma^2 \quad \Rightarrow \quad \sigma = \frac{a}{-\sqrt{1-k^2}}$$

Exercise 2: Compute the RBF by hand

Suppose our model function maps $\mathbb{R}^3$ to $\mathbb{R}$, and suppose we have two centers: $[1,0,-1]^T$ and $[1,1,0]^T$. Given the transfer function $\phi(r) = r^3$, and a set of weights $[1,2]$ and a bias (constant) of -1, use a calculator to compute the output of the RBF given the point $[-1,1,2]^T$.

SOLUTION: Given $\mathbf{x} = [-1,1,2]^T$, first compute the distances to the centers:

$$\left\| \begin{bmatrix} 1--1 \\ 0-1 \\ -1-2 \end{bmatrix} \right\| = \sqrt{2^2+1^2+3^2} = \sqrt{14} \qquad \left\| \begin{bmatrix} 1--1 \\ 1-1 \\ 0-2 \end{bmatrix} \right\| = \sqrt{2^2+0^2+2^2} = \sqrt{8}$$

Apply ϕ:

$$\phi(\|\mathbf{x}-\mathbf{c}_1\|) = 14^{3/2} \qquad \phi(\|\mathbf{x}-\mathbf{c}_2\|) = 8^{3/2}$$

So the output is:

$$w_1\phi(\|\mathbf{x}-\mathbf{c}_1\|) + w_1\phi(\|\mathbf{x}-\mathbf{c}_1\|) + b = 14^{3/2} + 2\,8^{3/2} - 1 \approx 96.638$$