Review Material, Exam 2

Here is a quick listing of the topics we have discussed since Exam 1:

- Projections
- The SVD
- How to determine the rank using the SVD.
- How to compute the low rank approximation to a data set (aka low dimensional representation) using the SVD.
- The pseudo-inverse (dagger)
- Using Matlab's slash command.
- Polynomial interpolation and the Vandermonde matrix.
- The Euclidean Distance Matrix (and interpolation using the EDM).
- The Radial Basis Function.
- Matlab data structures
- Gram-Schmidt
- Matlab training of the RBF using Orthogonal Least Squares (OLS).

Partial list of skills for the in-class portion

1. Be able to compute a projection matrix that projects a vector to another vector, and a vector into a subspace spanned by a given set of vectors. Be able to compute the component of a vector that is orthogonal to a given vector (see Gram Schmidt and OLS).
2. Use Gram-Schmidt to convert a given set of vectors to an orthonormal set.
3. Be able to compute the rank using a percentage of "total energy".
4. Be able to compute the low dimensional representation of a vector.
5. Be able to compute the pseudo-inverse using the SVD.
6. Be able to compute the EDM.
7. Be able to write the system of equations that we need to solve for the coefficients if we want a: (i) Best fitting polynomial, (ii) Best fitting model equation (linear in the unknowns), (iii) Interpolating polynomial, (iv) Interpolation with the EDM.
8. Be able to write the RBF as a function of $\mathbf{x}$.

Sample Questions

1. Compute the SVD of the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

2. Given that the SVD of a matrix was given in Matlab as:

```
>> [U,S,V]=svd(A)
U =
   -0.4346   -0.3010    0.7745    0.3326   -0.1000
   -0.1933   -0.3934    0.1103   -0.8886   -0.0777
    0.5484    0.5071    0.6045   -0.2605   -0.0944
    0.6715   -0.6841    0.0061    0.1770   -0.2231
    0.1488   -0.1720    0.1502   -0.0217    0.9619
S =
    5.72         0          0
         0       2.89       0
         0          0       0
         0          0       0
         0          0       0
V =
    0.2321   -0.9483    0.2166
   -0.2770    0.1490    0.9493
    0.9324    0.2803    0.2281
```

   (a) Which columns form a basis for the null space of A? For the column space of A? For the row space of A?

   (b) How do we "normalize" the singular values? In this case, what are they (numerically)?

   (c) What is the rank of A?

   (d) How would you compute the pseudo-inverse of A (do not actually do it):

   (e) Let B be formed using the first two columns of U. Would the matrix $B^T B$ have any special meaning? Would BB^T?

3. Suppose we have a subspace W spanned by an orthonormal set of non-zero vectors, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, each is in $\mathbb{R}^1 00$. If a vector $\mathbf{x}$ is in W, then there is a low dimensional (three dimensional in fact) representation of $\mathbf{x}$. What is it?

4. What is an RBF? Explain it so that a fellow student could follow your reasoning.

5. In Matlab, what does it mean to "train" an RBF? What does it mean to "simulate" an RBF?

6. Finish the definition: U is an orthogonal matrix if:

7. Finish the definition: Let P be $n \times n$. Then P is a projector if:

8. Finish the definition: Let P be $n \times n$. Then P is an orthogonal projector if:

9. If P is a projector, could P be invertible? Explain.

10. What is the orthogonal projector to the space spanned by a single vector $\mathbf{a}$?

11. Let the matrix X be given below- It has 3 points in $\mathbb{R}^2$.

$$X = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

    (a) Compute the EDM.

    (b) Using all the data as "centers", compute the matrix Φ, where

$$\Phi_{ij} = \phi(\|\mathbf{x}_j - \mathbf{c}_i\|)$$

    and $\phi(r) = r^3$.

    (c) What size would Y need to be, and what size would the matrix W be if we solve $W\Phi = Y$ for W? What if we wanted to include a bias (or constant) term?

12. What is the Gram-Schmidt algorithm? Give an example using

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

    HINT: When you compute these by hand, you don't need to normalize them.

13. Given the vector $\mathbf{x} = [1, 1, 1]^T$, which vector

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

    "best lies in the direction of $\mathbf{x}$" (in the sense of the subspace spanned by the vector, as in the OLS).

14. Let the matrix X store 4 points in $\mathbb{R}^2$, below:

$$X = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

    (a) Write the system of equations to find the polynomial, $p(x)$, that interpolates the data.

    (b) If we're looking for a function that maps the elements of the first row to the second row, what would the form of the RBF be (using the first three points as centers)? Write it out-