Chapter 1

The Basics

Against the disease of writing one must take special precautions, since it is a dangerous and contagious disease.

Peter Abelard
Letter 8, Abelard to Heloise

Judge an artist not by the quality of what is framed and hanging on the walls, but by the quality of what's in the wastebasket.

Anon., quoted by Leslie Lamport

It matters not how strait the gate,
How charged with punishments the scroll,
I am the master of my fate;
I am the captain of my soul.

W. E. Henley

Your manuscript is both good and original; but the part that is good is not original, and the part that is original is not good.

Samuel Johnson

In America only the successful writer is important, in France all writers are important, in England no writer is important, and in Australia you have to explain what a writer is.

Geoffrey Cotterel

It may be true that people who are merely mathematicians have certain specific shortcomings; however, that is not the fault of mathematics, but is true of every exclusive occupation.

Carl Friedrich Gauss
letter to H. C. Schumacher [1845]

In fifty years nobody will have tenure but everyone will have a Ph.D.

V. Wickerhauser

1.1 What It Is All About

In order to write effectively and well, you must have something to say. This sounds trite, but it is the single most important fact about writing. In order to write effectively and well, you also must have an audience. And you must *know consciously* who that audience is. Much of the bad writing that exists is performed by the author of a research paper who thinks that all his/her readers are Henri Poincaré, or by the author of a textbook who does not seem to realize that his/her readers will be students.

Good writing requires a certain confidence. You must be confident that you have something to say, and that that something is worth saying. But you also must have the confidence to know that "My audience is X and I will write for X." Many a writer of a mathematical paper seems to be writing primarily to convince himself that his/her theorem is correct, but not as an effort to communicate. Such an author is embarrassed to explain anything, and hides behind the details. Many a textbook author seems to be embarrassed to speak to the student in language that the student will apprehend. Such an author instead finds himself making excuses to the instructor (who either will not read the book, or will flip through it impatiently and entirely miss the author's efforts).

Imagine penning a poem to your one true love, all the while thinking "What would my English teacher think?" or "What would my pastor think?" or "What would my mother think?" Have the courage of your convictions. Speak to that person or to those people whom you are genuinely trying to reach. Know what it is that you want to say and then say it.

For a mathematician, the most important writing is the writing of a research paper. You have proved a nice theorem, perhaps a great theorem. You certainly have something to say. You also know exactly who your audience is: other research mathematicians who are interested in your field of study. Thus two of the biggest problems for a writer are already solved. The issue that remains is how to say it. Remember that, if you pen a love letter to yourself, then it will have both the good features and the bad features of such an exercise: it will exhibit both passion and fervor, but it will tend to exclude the rest of the world. What do these remarks mean in practice? In particular, they mean that as you write you must think of your reader—not yourself. You must consider his/her convenience, and his/her understanding—not your own.

In the Sputnik era, some years ago, when mathematics departments and journals were growing explosively and everyone was in a rush to publish, it was common to begin a paper by writing "Notation is as in my last paper." Today, by contrast, there are truly gifted mathematicians who write papers that look like a letter home to Mom: they just start to write, occasionally starting a new paragraph when the text spills over onto a new page, never formally stating a theorem or even a definition, never coming to any particular point. The contents are divine, if only the reader can screw up the courage to pry them loose.

These last are not the sorts of papers that you would want to read, so why torment your readers like this? Much of the remainder of this book will discuss ways to write your work so that people *will* want to read it, and will enjoy it when they do.

1.2 Who Is My Audience?

If you are writing a diary, then it may be safe to say that your audience is just yourself. (Truthfully, even this may not be accurate, for you may have it in the back of your mind that—like Anne Frank's diary, or Samuel Pepys's diary—this piece of writing is something for the ages.) If you are writing a letter home to Mom, then your audience is Mom and, on a good day, perhaps Pop. If you are writing a calculus exam, then your audience consists of your students, and perhaps some of your colleagues (or your chairman, if the chair is in the habit of reviewing your teaching). If you are writing a tract on handle-body theory, then your audience is probably a well-defined group of fellow mathematicians (most likely topologists). Know your audience!

Keep in mind a specific person—somebody actually in your acquaintance—to whom you might be writing. If you are writing to yourself or to Mom, this is easy. If you are instead writing to your peers in handlebody theory, then think of someone in particular—someone to whom you could be explaining your ideas. This technique is more than a facile artifice; it helps you to picture what questions might be asked, or what confusions might arise, or which details you might need to trot out and explain. It enables you to formulate the explanation of an idea, or the clarification of a difficult point.

I cannot repeat too often this fundamental dictum: have something to say and know what it is; know *why* you are saying it; finally, know to whom you are saying it, and keep that audience always in mind.

1.3 Writing and Thought

The ability to think clearly and the ability to write clearly are inextricably linked. If you cannot articulate a thought, formulate an argument, marshal data, assimilate ideas, organize a thesis, then you will not be an effective writer. By the same token, you can use your writing as a method of developing and honing your thoughts (see [Hig] for an insightful discussion of this concept).

We all know that one way to work out our thoughts is to engage in an animated discussion with someone whom we respect. But you can instead, à la Descartes, have that discussion with yourself. And a useful way to do so is by writing. When I want to work out my thoughts on some topic—teaching reform, or the funding of mathematics, or the directions that future research in several complex variables ought to take, or my new ideas about domains with noncompact automorphism group—I often find it useful to write a little essay on the subject. For writing forces me to express my ideas clearly and in the proper order, to fill in logical gaps, to sort out hypotheses from blind assumptions from conclusions, and to make my point forcefully and clearly. Sometimes I show the resulting essay to friends and colleagues, and sometimes I try to publish it. But, just as often, I file it away on my hard disc and forget it until I have future need to refer to it.

The writing of research level mathematics is a more formal process than that described in the last paragraph, but it can serve you just as well. When you write up your latest ideas for dissemination and publication, then you must finally face the music. That "obvious lemma" must now be treated; the case that you did not really want to consider must be dispatched. The ideas must be put in logical order and the chain of reasoning forged and fixed. It can be a real pleasure to craft your latest burst of creativity into a compelling flood of logic and calculation. In any event, this skill is one that you are obliged to master if you wish to see your work in print, and read by other people, and understood and appreciated.

Once you apprehend the principles just enunciated, writing ceases to be a dreary chore and instead turns into a constructive activity. It becomes a new challenge that you can aim to perfect—like your tennis backhand or your piano playing. If you are the sort of person who sits in front of the computer screen befuddled, frustrated, or even angry, and thinks "I know just what my thoughts are, but I cannot figure out how to say them" then something is wrong. Writing should *enable* you to express your thoughts, not hinder you. I hope that reading this book will help you to write, indeed will enable you to write, both effectively and well.

1.4 Say What You Mean; Mean What You Say

You have likely often heard, or perhaps uttered, a sentence like

> As a valued customer of XYZ Co., your call is very important to us. ✠

Or perhaps

> To assist you better, please select one of the following from our menu: ✠

What is wrong with these sentences? The first suggests that "your call" is a valued customer. Clearly that is not what was intended. A more accurately formulated sentence would be

> You are a valued customer of XYZ Co., and your call is very important to us.

or perhaps

> Because you are a valued customer of XYZ Co., your call is very important to us.

In the second example, the phrase "To assist you better" is clearly intended to modify "we" (that is, it is "we" who wish to better assist you); therefore a stronger construction is

> So that we may assist you better, please select an item from our menu ....

or perhaps

> We can assist you more efficiently if you will make a selection from the following menu.

What is the point here? Is this just pompous nit-picking? Assuredly not. Mathematics cannot tolerate imprecision. The nature of mathematical *notation* is that it tends to rule out imprecision. But the *words* that connect our formulae are also important. In the two examples given above, we may easily discern what the speaker intended; but, in mathematics, if you formulate your thoughts incorrectly then your point may well be lost. Here are a few more examples of sentences that do not convey what their authors intended:

> Having spoken at hundreds of universities, the brontosaurus was a large green lizard. ✠

(Amazingly, this sentence is a slight variant of one that was uttered by a distinguished scholar who is world famous for his careful use of prose.)

> As in our food, we strive to be creative with keeping the highest quality in mind, we have in our wine selections also. ✠

(This sentence was taken from the menu of a rather good St. Louis restaurant.)

> To serve you better, please form a line. ✠

(How many times have you heard this at your local retailer's, or at the bank?)

The message here is a simple one: Make sure that your subject matches your verb. Make sure that your referents actually refer to the person or thing that is intended. Make sure that your participles do not dangle. Make sure that your clauses cohere. *Read each sentence aloud*. Does each one make sense? Would you *say this in a conversation?* Would you understand it if someone else said it?

Use words carefully. A well-trained mathematician is not likely to use the word "continuous" to mean "measurable" nor "convex" to mean "one-connected". However we sometimes lapse into sloppiness when using ordinary prose. Treat your dictionary as a close friend: consult it frequently. As a consequence, do not use "enervate" to mean "invigorate" nor "fatuous" to mean "overweight" nor "provenance" to denote a geopolitical entity. When I am being underhanded, it is not because I am short of help.

In life, we receive many different streams of ideas simultaneously, and we parallel-process them in that greatest of all CPUs—the human brain. We absorb and process information and knowledge in a nonlinear fashion. But written discourse is linearly ordered. Word k proceeds directly after word $(k-1)$. The distinction between written language as a medium and the information flow that we commonly experience is one of the barriers between you and good writing. As you read this book (which purports to tell you how to write), you will see passages in which I say "now I will digress for a moment" or "here is an aside." (In other places I put sentences in parentheses or brackets; or I use a footnote.) These are junctures at which I could not fit the material being discussed into strictly logical order. You will have to learn to wrestle with similar problems in your own writing. One version of writer's block is a congenital inability to address this linear vs. nonlinear problem. In this situation, nothing succeeds like success. I recommend that, next time you encounter this difficulty, address it head on. After you have defeated it a few times (not without a struggle!), then you will be confident that you can handle it in the future.

I have discoursed on accurate use of language in the technical sense. Now let me remark on more global issues. As you write, you must think not only about whether your writing is correct and appropriate, but also about where your writing will go and what it will do when it gets there.

I have already admonished you to know when to start writing. Namely, you begin writing when you have something to say and you know clearly to whom you wish to say it. You also must know when to stop writing. Stop when you have said what you have to say. Say it clearly, say it completely, say it forcefully, say it without leap or lacuna, but then shut up. To prattle on and on is not to convince further.

And never doubt that language is a weapon. "Sticks and stones may break my bones but words will never hurt me" is perhaps the most foolish sentence ever uttered. You can inflict more damage, more permanently, with words than you can with any weapon. You can manipulate more minds, and more people, with words than with any other device. For example, when journalists in the 1960s referred to "self-styled radical leader Abbie Hoffman", they downgraded Hoffman in people's minds. They never referred to Spiro Agnew as a "self-styled [you fill in the blank]" or to Gordon Liddy as a "self-styled ...". This moniker was reserved for Abbie Hoffman—and sometimes for Jerry Rubin and Mario Savio—and one cannot help but surmise that it was for a reason. By the same token, newspapers frequently spoke in the 1960s of "outside agitators" visiting university campuses. They were never described as "colloquium speakers" or "expert political consultants".

When a policeman addresses you by

> Sir, may I see your driver's license? Did you notice that red light back there?

then he is sending out one sort of signal. (Namely, you are clearly a law-abiding citizen and he is just doing his/her job by pulling you over and perhaps giving you a ticket.) When instead a cop in the station house says

> OK, Billy. Why don't you spill your guts? You know that those other bums aren't going to do a thing to protect you. All they care about is saving their own skins. Jacko already confessed to the heist and told us that you held the gun, Billy. Now we need to hear it from you. Make it easy on yourself, Billy: play ball with us and we'll play ball with you.

then he is sending out a different sort of signal. (Namely, by using the first name—and not "William", but "Billy"—he is undercutting the addressee's dignity; he is treating the person like a wayward child. Further, the policeman is cutting off the individual from his/her peers, making him feel as though he is on his/her own. He is suggesting—albeit vaguely—that he may be willing to cut a deal.)

When you are a person of some accomplishment, and some clout, then your writing carries considerable responsibility. Your words may have great effect. You must weigh the words, and weigh their impact, carefully.

I am going to conclude this section with a brief homily. (I promise that there will be no additional homilies in the book; you may even ignore this one if you wish.) Nikolai Lenin said that the most effective way to bring down a society is to corrupt its language.[1] You need only look around you to perceive the truth of this statement. When language is corrupted, then people do not communicate effectively. When they do not communicate effectively, then they cannot cooperate. When they cannot cooperate, then the fabric of civilization begins to unravel.

Some of us use the word "bad" to mean "good." We use the phrase "let us keep our neighborhoods safe and clean" to mean "let us segregate our schools and arm every home"; we use the phrase "I am for gun control and freedom of choice" to mean "I'm a liberal and you're not." We say "account executive" when we mean "sales clerk" and "sanitation engineer" when we mean "garbage man." We use the words "interesting" to mean "foolish," "imaginative" to mean "irresponsible," and "naive" to mean "idiotic." These observations are not just idle cocktail party banter. They are in fact indicative of barriers between certain social groups.

[1] A similar statement is attributed to John Locke in *On Human Understanding.*

It is just the same in mathematics. When we use the word "proof" to mean "guesses based on computer printouts" (see [Hor]), when we use "theoretical mathematics" to mean "speculative mathematics" (see [JQ]), when we use the phrase "Charles mathematicians" to belittle the practitioners of traditional and hard-won modes of reasoning that have been developed over many centuries (see [Ati, pp. 193–196]), when we use the phrase "new mathematics" to mean "facile intuition" (see [PS], [Ati, pp. 193–196]), then we are corrupting our subject. These are gross examples, but the same type of corruption occurs in the small when we write our work sloppily or not at all. It is the responsibility of today's scholars to develop, nurture, and record our subject for future generations. Good writing is of course not an end in itself; writing is instead the means for achieving the important goal of communicating and preserving mathematics.

1.5 Proofreading, Reading for Sound, Reading for Sense

Proofreading is an essential part of the writing process. And it is not a trivial one. You do not simply write the words and then quickly scan them to be sure that there are no gross errors. Paul Halmos [Hig] said that he never published a word before he had read it six times. Not all of us are that careful, but the spirit of his practice is correct:

- One proofreading should be to check *spelling* and simple *syntax* errors (software can help with the former, and even with the latter—see Section 6.4).
- One proofreading should be for *accuracy*.
- One proofreading should be for *organization* and for *logic*.
- One proofreading should be for *sense*, and for the flow of the ideas.
- One proofreading should be for *sound*.
- One proofreading should be for overall coherence.

The great English stage actor Laurence Olivier used to rehearse Shakespeare by striding across the countryside and delivering his lines to herds of bewildered cattle. Understandably, you may be disinclined to emulate this practice when developing your next paper on p-adic L functions—especially if you live in Brooklyn. However, note this: all the best writers whom I know read their work aloud to themselves. Reading your words aloud *forces* you to make sense of what you have written, and to deliver it as a coherent whole. If you have never tried this technique, then your first experience with it will be a revelation. You will find that you quickly develop a new sensitivity for sound and sense in your writing. You will develop an "ear." You will learn instinctively what works and what does not.

Consider these simple examples. Suppose that the Hemingway novel *For Whom the Bell Tolls* were instead entitled *Who the Dingdong Rings For*; or that the Thornton Wilder play *Our Town* were called *My Turf*. Even though the sense of the titles has not been changed appreciably, we see that the alternative titles eschew all the poetry and imagery that is present in the originals. *For Whom the Bell Tolls* evokes powerful emotions; the proffered alternative falls flat. The title *Our Town* suggests one value system, while *My Turf* brings to mind another. One fancies that, if *The Scarlet Letter* had had a less poetic title (how about *Bad Girls Finish Last*), then perhaps Hester Prynne would have garnered only an "$A-$," or maybe even an "Incomplete."

Mathematicians rarely have to wrestle with these poetic questions. But we need to choose names for mathematical objects; we need to formulate definitions. We need to describe and to explain. My Ph.D. advisor thought very carefully about his choice of notation and choice of terminology. He figured that his ideas would have considerable influence and lasting value, and he wanted them to come out right.

As an instance of these ideas, the word "continuous" is a perfect name for a certain type of function; the alternative terminology "non-hypererratic" would be much less useful. The phrase "the point x lies in a relative neighborhood of P" conveys a world of meaning in an elegant and memorable fashion. Not by accident has this terminology become universal. You should strive for this type of precision and elegance in your own writing.

William Shakespeare said that "... a rose by any other name would smell as sweet." This statement is true, and an apt observation, in the context of the dilemma that faced Romeo and Juliet. But the name of a person, place, or thing can profoundly affect its future. There will never be a great romantic leading man of stage and screen who is named Eggs Benedict and there will never be a Fields Medalist or other eminent mathematician named Turkey Tetrazzini. The name of an object does not change its properties (consult Saul Kripke's New Theory of Reference for more on this thought), but it can change the way that the object is perceived by the world at large. Bear this notion in mind as you create terminology, formulate definitions, and give titles to your papers and other works.

Have you ever noticed that, when you are reading a menu or listening to an advertisement, it never fails that the food being described contains "fresh creamery butter" and "pure golden honey"? The marketing people never say "this grub contains butter and honey," for there is nothing appealing about the latter statement. But the first two evoke images of delicious food. As mathematicians, we are not in the position of hawking victuals. But we still must make choices to convey most effectively a given message, and the spirit of that message. We want to inform, and also to inspire. Consider the sentence

> The conjecture of Gauss (1830) is false. ✠

Contrast this rather bald statement with

> The lemmas of Euler (1766) and the example of Abel (1827) led Gauss to conjecture (1830) that all semistable curves are modular. The conjecture was widely believed, and more than fifty papers were written by Jacobi, Dirichlet, and Galois in support of it. To everyone's surprise and dismay, a counterexample was produced by Frobenius in 1902. This counterexample opened many doors.

There is no denying that the second passage puts the entire matter in context, tells the reader who worked on the conjecture and why, and also how the matter was finally resolved. There is a tradition in written mathematics to conform to the terse. In your own writing, consider instead the advantages of telling the reader what is going on.

My advice is not to agonize over each word as you write a first draft. Just get the ideas down on the page. But *do* agonize a bit when you are editing and proofreading. A passage that reads

> This is a very important operator, that has very specific properties, culminating in a very significant theorem. ✠

is all right as a first try, but does not work well in the long run. It overuses the word "very." It does not flow smoothly. It makes the writer sound dull witted. Consider instead

> This operator will be significant for our studies. Its spectral properties, together with the fact that it is smoothing of order one, will lead to our first fundamental theorem.

The second passage differs from the first in that it has *content*. It says something. It flows nicely, and makes the writer sound as though he/she has something worthwhile to offer.

An amusing piece of advice, taken from [KnLR, p. 102], is never to use "very" unless you would be comfortable using "damn" in its place.

A good, though not ironclad, rule of thumb is not to use the same word, nor even the same sound, in two consecutive sentences. Of course you may reuse the word "the," and the nouns that you are discussing will certainly be repeated; but, if possible, do not repeat descriptive words and do not place words that sound similar in close proximity.

Also be careful of alliteration. Vice President Spiro Agnew, with the help of speech writer William Safire, earned for himself a certain reputation by using phrases like "pampered prodigies," "pusillanimous pussyfooters," "vicars of vacillation," and "nattering nabobs of negativism." Whatever élan accrued to Agnew by way of this device is probably not something that you wish to cultivate for yourself. Lyndon Johnson led us into an escalated Vietnam war by deriding "nervous nellies." The alliterative device is often suitable for polemicism or poetry, but rarely so for mathematics. For example

> This semisimple, sesquilinear operator serves to show sometimes that subgroups of S are sequenced. ✠

does not sound like mathematics. The typical reader probably will pause, reread the sentence several times, and wonder whether the writer is putting him/her on. Better is

> Observe that this operator is both semisimple and sesquilinear. These properties can lead to the conclusion that if G is a subgroup of S then G is sequenced.

Notice how simple syntactical tricks are used to break up the alliteration, and to good effect.

The last two points—not to repeat words or sounds, and to avoid intrusive alliteration—illustrate the principle of "sound and sense." If you read your work aloud as you edit and revise, then you will pick out offending passages quickly and easily. With practice, you also will learn how to repair them. The result will be clearer, more effective writing.

1.6 Compound Sentences, Passive Voice

It would be splendid if we could all write with the artistry of Flaubert, the elegance of Shakespeare, and the wisdom of Goethe. In mathematical writing, however, such an abundance of talent is neither necessary nor called for. In developing an intuitionistic ethics ([Moo]), for example, one presents the ideas as part of a ritualistic dance: there is a certain intellectual pageantry that comes with the territory. In mathematics, what is needed is a clear and orderly presentation of the ideas.

Mathematics is already, by its nature, logically complex and subtle. The sentences that link the mathematics are usually most effective when they are simple, declarative sentences. Compound sentences should be broken up into simple sentences. Avoid run-on sentences at all cost. Here are some examples:

Rather than saying

> As we let x become closer and closer to 0, then y tends ever closer to t_0. ✠

instead say

> When x is close to 0 then y is close to t_0.

Of course mathematical notation allows us to write $\lim_{x \to 0} y = t_0$ instead of either of these; this abbreviated presentation will, in many contexts, be more desirable.

Rather than saying

> If g is positive, f is continuous, the domain of f is open, and we further invoke Lemma 2.3.6, then the set of points at which $f \cdot g$ is differentiable is a set of the second category, provided that the space of definition of f is metrizable and separable. ✠

instead say

> Let X be a separable metric space. Let f be a continuous function that is defined on an open subset of X. Suppose that g is any positive function. Using Lemma 2.3.6, we see that the set of points at which $f \cdot g$ is differentiable is of second category.

An alternative formulation, even clearer, is this:

> Let X be a separable metric space. Let f be a continuous function that is defined on an open subset of X. Suppose that g is any positive function. Define S to be the set of points x such that $f \cdot g$ is differentiable at x. Then, by Lemma 2.3.6, S is of second category.

Note the use of the words "suppose" and "define" to break up the monotony of "let." Observe how the formal definition of the set S clarifies the slightly awkward construction in the penultimate version of our statement.

It is tempting, indeed it is a trap that we all fall into, to overuse a single word that means "hence" or "therefore." An experienced mathematical writer will have a clutch of words (such as "thus," "so," "it follows that," "as a result," and so on) to use instead. A paragraph in which every sentence begins with "therefore," or with "let," or "so" can be uncomfortable to read. Have alternatives at your fingertips.

In general, you should avoid introducing unnecessary notation. Mary Ellen Rudin's famous statement

> Let X be a set. Call it Y.

is funny because it is so ludicrous. But this example is not far from the way we write when we are seduced by notation. A statement like

> Let X be a compact metric subspace of the space Y. If f is a continuous, $\mathbb{R}$-valued function on that space then it assumes both a maximum and a minimum value. ✠

suffers from giving names to the metric space, its superspace, the function, and the target space, and then never using any of them. Slightly better is

> Let X be a compact metric space. If f is a continuous, real-valued function on X then f assumes both a maximum and a minimum value.

Better still is

> A continuous, real-valued function on a compact metric space assumes both a maximum value and a minimum value.

The last version of the statement uses no notation, yet conveys the message both succinctly and clearly.

Paul Halmos [Ste] asserts that mathematics should be written so that it reads like a conversation between two mathematicians who are on a walk in the woods. The implementation of this advice may require some effort. If what you have in mind is a huge commutative diagram, or the determinant of a big matrix whose entries are all functions, then you will likely be unsuccessful in conveying your thoughts orally. You must think in terms of how you, or another reasonable person, would *understand* such a complicated object. Of course such understanding is achieved in bits and pieces, and it is achieved conceptually. That is how you will communicate your ideas during a walk in the woods.

One corollary of the "walk in the woods" approach to writing is that you should write for a reader who is not necessarily sitting in a library, with all the necessary references at his/her fingertips. To be sure, most any reader will have to look up a few things. But if the reader must race to the stacks, or boot up the computer and do a `Google` search, at every other sentence, then you are making the job too hard. Your paper is far too difficult to follow. Supply the necessary detail, and the proper heuristic, so that even if the reader is not sure of a notion he/she will be able temporarily to suspend his/her disbelief and move on.

Most authorities believe that writing in the passive voice is less effective than writing in the active voice. To write in the active voice is to identify the agent of the action, and to emphasize that agent (see [Dup] for a powerful discussion of active voice vs. passive voice). For example,

> The manifold M is acted upon by the Lie group G as follows: ✠

is less direct, and requires more words, than

> The Lie group G acts on the manifold M as follows:

Likewise, the statement

> It follows that the set Z will have no element of the set Y lying in it. ✠

can be more clearly expressed as

> Therefore no element of Y lies in Z.

Even better is

> The sets Y and Z are disjoint.

or

> Therefore $Y \cap Z = \emptyset$.

Notice that the last version of the statement used one word, while the first version used fifteen. Also, a mathematician much more readily apprehends $Y \cap Z = \emptyset$ than he/she does a string of verbiage. Finally, coming up with the succinct fourth formulation required not only restating the proposition, but also thinking about its meaning. The result was plainly worth the effort.

In spite of these examples, and my warnings against passive voice, I must admit that passive voice gives us certain latitude that we do not want to forfeit. If, in the first example, you have reason to stress the role of the manifold M over the Lie group G, then you may wish to use passive voice. In the second example, it is unclear how the use of passive voice could add a useful nuance to your thoughts. As usual, you must let sound work with sense to convey your message.

As I have already noted, no rule of writing is unbreakable. The active voice is usually more effective than the passive voice. It is easy to criticize Lincoln's Gettysburg address for over-use of the passive voice. But Lincoln had a good ear. If he had begun the speech with

> Our ancestors founded this country 87 years ago.

then he would have certainly followed the dictum of using the active voice and using simple declarative sentences. However he would not have set the beautiful pace and tone that "Fourscore and seven years ago our fathers brought forth on this continent, a new nation, ..." invokes. He would have jumped too quickly into the rather difficult subject matter of his speech. (See [SW] for the provenance of these last ideas.)

As mathematicians, we rarely will be faced with a choice analogous to Lincoln's. But the principle illustrated here is one worth appreciating.

1.7 Technical Aspects of Writing a Paper

Even when your paper is in draft form, your name should be on it. A date is helpful as well. Number the pages. Write on one side of the paper only. Give the paper a working title.

Is all this just too compulsive? No.

First, you must always put your name on your work to identify it as your own. If it contains a good idea, then you do not want someone else to walk off with it. Because you tend to generate so many different drafts and versions of the things that you write, you should date your work. I have even known mathematicians who put a time of day on each draft. (Of course a computer puts a date and time stamp on each *computer file* automatically; here I am discussing hard copy or paper drafts.)

You should write your affiliation—even on the draft. If you are usually at Harvard, then write that down. If instead you are spending the year in Princeton, write that down. The draft could, at some point, be circulated. People need to know where to find you. With this notion in mind, include your current *e*-mail address.

If your writing is highly technical, and you are deeply involved in working out a complicated idea, then you do not want to burden yourself with not knowing in which order the pages go. Be sure to number them. The numbering system need not be "1 2 3 4 5 . . .." It could be "$1A$ $1B$ $1C$ . . ." or "$1_{\rm cov}$ $2_{\rm cov}$ $3_{\rm cov}$ . . . " (to denote your subsection on the all-important covering lemma). In a rough draft, self-serving numbering systems like these can be marvelously useful. On the preprint that you intend to circulate, use a traditional sequential method for numbering the pages.

Take a few moments to think about the numbering of theorems, definitions, and so forth. This task is important both in writing a paper and in writing a book. Some authors number their theorems from 1 to n, their definitions from 1 to k, their lemmas from 1 to p, their corollaries from 1 to r—each item having its own numbering system. Do not laugh: this describes the default in LaTeX. As a reader, I find this method maddening; for the upshot is that I can never find anything. For instance, if I am on the page that contains Lemma 1.6, then that gives me no clue about where to find Theorem 1.5. If, instead, all displayed items are numbered in sequence—Theorem 1.2 followed by Corollary 1.3 followed by Definition 1.4, etc.—then I always know where I am.

Having decided on the logic of your numbering system, you also need to decide how much information you want each number to contain. What does this mean? My favorite numbering system (in writing a book) is to let "⟨⟨*Item*⟩⟩ 3.6.4" denote the fourth displayed item (theorem or corollary or lemma or definition) in the sixth section of Chapter 3. If there is a labeled, displayed equation in the statement of the ⟨⟨*Item*⟩⟩ then I label it (3.6.4.1). The good feature of this system is that the reader always knows precisely where he/she is, and can find anything easily. The bad feature is that the numbering system is a bit cumbersome. Other authors prefer to number displayed items within each section. Thus, in Section 6 of Chapter 3 the displayed items are numbered simply 1, 2, 3, ... When reference is later made to a theorem, the reference is phrased as "by Theorem 4 in Section 6 of Chapter 3" or "by Theorem 4 of Section 3.6." As you can see, this ostensibly simpler numbering system is cumbersome in its own fashion.

The main point is that you want to choose a numbering system that suits your purposes, and to use it consistently. You want to make your book or paper as easy as possible for your reader to study. Achieving this end requires that you attend to many small details. Your numbering system is one of the most important of these.

A final point is this: do not number every single thing in your manuscript. This dictum applies whether you are writing a paper or a book. I have seen mathematical writing in which every single paragraph is numbered. Such a device certainly makes navigation easy. But it is cumbersome beyond belief. Likewise do not number all formulas. You will only be referring to some of them, and the reader knows that. To number all formulas will create confusion in the reader's mind; he/she will no longer be able to discern what is important and what is less so.

When writing your draft (by hand), write on one side of the paper only. If you do not, and if you are writing something fairly technical and complicated (like mathematics), then you can become hopelessly confused when trying to find your place. In addition, you must frequently set two pages side by side—for the sake of comparing formulas, for instance. This move is easy with a manuscript written on one side, and nearly impossible with one that is not.

If you are scrupulous about not wasting paper, and insist on using both sides, then my advice is this: write drafts of your mathematical papers on one side of fresh paper. When that work is typed up and out the door, boldly X- out the writing on the front side of each page of your old drafts. Turn the paper over, and use it as scratch paper, or for your laundry list.

I suggest writing in ink. Pencil can smear, erasing can tear the page, and it is difficult to read a palimpsest. Also pencil-written material does not photocopy well. Blue pens do not photocopy well either. I always write with a black pen on either white or yellow paper. I write either with a fountain pen or a rolling writer or a fiber-tip pen so that the pen strokes are *dense* and *sharp* and *dark*. I write with a pen that does not skip or blot. If it begins to do either, I immediately discard it and grab a new one.

Of course you cannot erase words that are written with a pen; but you can cross them out, and that is much cleaner. It is easier to read a page written in bold black ink, and which includes some crossed out passages, than to decipher a page of chicken scratch layered over erased smears written with a pencil or written with a pen that is not working properly.

Be sure that your desk is well stocked with paper, pens, Wite-Out®, Post-it® notes, a stapler, staples, a staple remover, cellophane tape, paper clips, manila folders, manila envelopes, scissors, a dictionary, and anything else you may need for writing. Have them all at your fingertips. You do not want to interrupt the precious writing process by running around and looking for something trivial.

Do not write much on each page. I advise writing *large*, and double or triple spaced. The reason? First, you want to be able to insert passages, make editorial remarks, make corrections, and so forth. Second, a page full of cramped writing on every line is hard to read. Third, you can more easily rearrange material if there is just a little on each page. For example, if one page contains the statement of the main theorem and nothing else, another contains key definitions and nothing else, and so forth, then you can easily change the location of the main theorem in the body of the paper. If the main theorem is buried in a page with a great deal of other material, then moving it would involve either copying, or photocopying, or cutting with scissors.

Do not hesitate to use colored pens. For instance, you could be writing text in black ink, making remarks and notes to yourself (like "find this reference" or "fill in this gap") in red ink, and marking unusual characters in green ink. This may sound compulsive, but it makes the editing process much easier.

A good bibliography is an important component of scholarly work (more on bibliographies can be found in Sections 2.6, 5.5). Suppose that you are writing a paper with a modest number of references (about 25, say), and you are assigning an acronym to each one. For instance, [GH] could refer to the famous book by Griffiths and Harris. When you refer to this work while you are writing, use the acronym. Keep a sheet of notes to remind yourself what each acronym denotes. Do not worry about looking up the detailed bibliographic reference while you are engaged in writing; instead, compartmentalize the procedure. When you are finished writing the paper, you will have a complete, *informal* list of all your references. You can go to `MathSciNet` (Section 7.2) OnLine and find most of your references in an instant. You can also go to your library's catalog OnLine to find locally obtainable references. LaTeX can be a great help in eliminating much of the tedium of assembling bibliographies. See the discussion in Sections 2.6 and 5.5.

Let me make a general remark about the writing process. As you are writing a paper, there will be several junctures at which you feel that you need to look something up: either you cannot remember a theorem, or you have lost a formula, or you need to imitate someone else's proof. My advice is *not* to interrupt yourself while you are writing. Take your red pen and make a note to yourself about what is needed. But *keep writing*. When you are in the mood to write, you should take advantage of the moment and do just that. Interrupting yourself to run to the library, or for any other reason, is a mistake.

Write on a desk that is free of clutter. It is romantic, to be sure, to watch a film in which the writer labors furiously on a desk that is awash with papers, books, hamburger bags, ice cream containers, old coffee cups, last week's underwear, and who knows what else. Leave that stuff to the movies. Instead imagine tearing into page 33 of your manuscript and accidentally spilling a week-old cup of coffee and a piece of pepperoni pizza all over your project. Think of the time lost in mopping up the mess, separating the pages, trying to read what you wrote, copying your pages, and so forth. Enough said.

If you are going to drink coffee or a soda or eat a sandwich while you work, I suggest having the food on a small separate side table. This little convenience will force you to be careful, and if you do have an accident then it will not make a mess of your work.

Write in a place where you can concentrate without interruption. Whether you have music going, or a white noise machine playing, or a strobe light flashing is your decision. But if you are going to concentrate on your mathematics, it may take up to an hour to get the wheels turning, to fill your head with all the ideas you need, and to start formulating the necessary assertions. After you have invested the necessary time to tool up, you want to use it effectively. Therefore you do not want to be interrupted. Close the door and unplug the telephone if you must. Victor Hugo used to remove all his clothes and have his servant lock him in a room with nothing but paper and a pen. Moreover, the servant guarded the door so that the great man would not be interrupted by so much as a knock. This method is not very practical, and is perhaps not well suited to modern living, but it is definitely in the right spirit.

1.8 More Specifics of Mathematical Writing

For the most part, the writing of mathematics is like the writing of English prose. Indeed, it *is* a part of the writing of English. (*Caveat:* I hope that my remarks have some universality, and apply even if you are writing mathematics in Tagalog or Coptic or Tlingit.) If you read your work aloud (I advocate this practice in Section 1.5), then you should be reading complete sentences that flow from one to the next, just as they do in good prose.

It is all too easy to write a passage like

> Look at this here equation:
>
> $$x^n + y^n = z^n. \qquad \maltese$$

Much smoother is the passage

> The equation
>
> $$x^n + y^n = z^n$$
>
> tells us that Fermat's Last Theorem is still alive.

Another example of good sentence structure is

> Since
> $$A < B$$
> we know that ....

Notice that the the sentence reads well aloud: "Since A is less than B we know that ...."

An aspect of writing that is peculiar to mathematics is the use of notation. Without good notation, many mathematical ideas would be difficult to express. Indeed, the development of mathematics in the middle ages and the early renaissance was hobbled by a lack of notation. With good notation, our writing has the potential to be elegant and compelling.

A common misuse of notation is to put it at the beginning of a sentence or a clause. For example,

> Let f be a function. f is said to be *semicontinuous* if ...
> ✠

and

> For most points x, $x \in S$. ✠

Even in these two simple examples you can begin to apprehend the problem: the eye balks at a sentence or clause that is begun with a symbol. You find yourself rereading the passage a couple of times in order to discern the correct sense. Much better is:

> A function f is said to be *semicontinuous* if ...

and

> We see that $x \in S$ for most points x.

Observe that both of these revisions are easily comprehended the first time through. That is one of the goals of good writing.

Mathematical notation is often so elegant and compelling that we are tempted to overuse, or misuse, it. For example, the notation in the sentence "If $x > 0$, then $x^2 > 0$" is no hindrance, is easy to read, and tends to make the sentence short and sweet (nonetheless, there are those who would tender cogent arguments for "If a number is positive then so is its square."). By contrast, the phrase

Every real, nonsquare $x < 0 \ldots$ ✠

is objectionable. The reason is that it is not clear, on a first reading, what is meant. Are you saying that "Every real, nonsquare x is negative" or are you saying "Every real, nonsquare x that is less than zero has the additional property ...." By strictest rules, the notation $<$ is a *binary connective*. The notation is designed for expressing the thought $A < B$. If that is not the exact phrase that fits into your sentence, then you had best not use this notation.

When you are planning a paper, or a book, you should try to plan your notation in advance. You want to be consistent throughout the work in question. To be sure, we have all seen works that, in Section 9, say "For convenience we now change notation." All of a sudden, the author stops using the letter H to denote a subgroup and instead begins to use H to denote a biholomorphic mapping. Amazingly, this abrupt device actually works much of the time—at least with professional mathematicians. But you should avoid it. If you can, use the same notation for a domain in Section 10 (or Chapter 10) of your work that you used in Section 1 (or Chapter 1). Try to avoid local contradictions—like suddenly shifting your free variable from x to y. Try not to use the same character for two different purposes.[2]

The last stipulation is not always easy to follow. Many of us commonly use i for the index of a series or sequence: $\sum_{i=1}^{\infty} a_i$ and a_i. No problem so far, but suppose that you are a complex analyst, and use i to denote a square root of -1. And now suppose that this last i occurs in some of your sequences and series. You can see the difficulties that would arise. It is probably best to use j or k as the index of your sequence or series. A little planning can help with this problem, though

[2]When André Weil was writing his book *Basic Number Theory* [Wei], he strove mightily to follow this advice. He used up all the roman letters, all the Greek letters, all the fraktur letters, all the script letters, all the Hebrew letters, and all the other commonly used characters that are seen in mathematics. He ended up resorting to Japanese characters.

in the end it may involve a great deal of tedious work to weed out all notational ambiguities.

Many a budding mathematician is seduced by mathematical notation. There was a stage in my education when I thought that all of mathematics should be written without words. I wrote long, convoluted streams of $\forall$, $\exists$, $\ni:$, $\Rightarrow$, $\equiv$, and so forth. This style would have served me well had I been invited to coauthor a new edition of *Principia Mathematica* (see [WR]). In modern mathematics, however, you should endeavor to use English—and to minimize the use of cumbersome notation. Why burden the reader with

$$\forall x \exists y, x \geq 0 \Rightarrow y^2 = x \qquad ✠$$

when you can instead say

> Every nonnegative real number has a square root.

The most important logical syllogism for the mathematician is *modus ponendo ponens*, or "if ... then." If you begin a sentence with the word "if," then do not forget to include the word "then." Consider this example:

> If $x > 4$, $y < 2$, the circle has radius at least 6, the sky is blue, the circle can be squared. ✠

Which part of this sentence is the hypothesis and which the conclusion? After a few readings you may be able to figure it out. If it were sensible mathematics then the mathematical meaning would probably give you some clues. But it is clearer to write

> If $x > 4$, $y < 2$, the circle has radius at least 6, and the sky is blue, then the circle can be squared.

Following the dictum that shorter sentences are frequently preferable to longer ones, you can express the preceding thought even more succinctly as

> Suppose that $x > 4$, $y < 2$, the circle has radius at least 6, and the sky is blue. Then the circle can be squared.

The word "then" is pivotal to the logical structure here. It acts both as a connective and as a sign post. The reader can (usually) figure out what is meant if the word "then" is omitted. But the reader should not *have* to do so. Your job as the writer is to perform this task *for* the reader. Mathematicians have a tendency to want to jam everything into one sentence. However, as the last example illustrates, greater clarity can often be achieved by breaking things up; this device also forces you to think more clearly and to organize your thoughts more effectively.

Mathematicians commonly write "If f is a continuous function, then prove X." A moment's thought shows that this is not the intended meaning: the desire to prove X is not contingent on the continuity of f. What is intended is "Prove that, if f is a continuous function, then X." In other words, the hypothesis about f is part of what needs to be proved.

Sometimes you need to write a sentence that treats a word as an object. Here is an example:

> We call Γ the *fundamental solution* for the partial differential operator L. We use the definite article "the" because, suitably normalized, there is only one fundamental solution.

I have oversimplified the mathematics here to make a typographical point. First, when you define a term (for the first time), you should italicize the word or phrase that is being defined. Second, when you refer to a word (in this case "the") as the object of discussion, then put that word in quotation marks. For a variety of psychological reasons, writers often do not follow this rule. It is helpful to recall W. V. O. Quine's admonition: " 'Boston' has six letters. However Boston has 6 million people and no letters."

The phrase "if and only if" is a useful mathematical device. It indicates logical equivalence of the two phrases that it connects. While the phrase is surely used in some other disciplines, it plays a special role in mathematical writing; we should take some care to treat it with deference. Some people choose to write it as "if, and only if,"—with two commas. That is perfectly grammatical, if a little stilted. One habit that is unacceptable (because it sounds artificial and is difficult to read) is to begin a sentence with this phrase. For instance,

> If and only if x is nonnegative, can we be sure that the real number x has a real square root. ✠

That is a painful sentence to read, whether the reading is done aloud or *sotto voce*. Better is

> A real number x has a real square root if and only if $x \geq 0$.

An alternative form, not with universal appeal (but better than beginning a sentence with "if and only if"), is

> Nonnegative real numbers, and only those, have real square roots.

The neologism "iff," reputed to have been popularized by Paul Halmos, is a generally accepted abbreviation for "if and only if." This is a useful bridge between the formality of "if and only if" and the convenience of "if." It is also common to use the symbol $\Longleftrightarrow$ for "if and only if."

Word order can have a serious, if subtle, effect on the meaning (or at least the nuance) of a sentence. The examples

> Yellow is the color of my true love's hair.

> My true love's hair has the color yellow.

> The hair, which is yellow, of my true love ...

each say something different, as they emphasize a different aspect—either the color, or the person, or the hair—that is being considered. (As an exercise, insert the word "only" into all possible positions in the sentence

> I helped Carl prove quadratic reciprocity last week.

and watch the meaning change.)

In mathematics, word order can seriously alter the meaning of a sentence, with the result that the sentence is not immediately understood—if at all. When you proofread your own work, you tend to supply meaning that is not actually present in the writing; the result is that you can easily miss obscurity imposed by word order. Reading your work aloud can help cut through the problem.

Do not overuse commas. I become physically ill when I see a sentence like

> We went to the store, to buy some potatoes. ✠

Slightly more subtle, but still irksome, is

> Now that we have our hypotheses in place, we state our theorem, with the point in mind, that we wish to understand the continuity, of functions in the class $\mathcal{S}$. ✠

We certainly use a comma to indicate a pause. But the comma indicates a *logical pause*, not a lack of air or lack of good sense. Read the last displayed sentence out loud, with suitable pauses where the commas occur. It sounds like someone huffing and puffing; the pauses have no reason to them. This sentence is not a representative example of the way that we speak, hence it is not indicative of the way that we should write. Much more attractive is

> Our hypotheses are now in place, and we next state our theorem. The point is to understand the continuity properties of functions belonging to the class $\mathcal{S}$.

Mathematicians like the word "given." We tend to overuse and misuse it—especially in instances where the word can be discarded entirely. Consider the example "Given a metric space X, and a point $p \in X$, we see that ...." More direct is "If X is a metric space and $p \in X$, then ...." We are often tempted to transcribe spoken language and call that written language; such laziness should be defeated. Our misuse of "given" is an example of such sloth.

Whenever possible, use singular constructions rather than plural. Consider the sentence

> Domains with noncompact automorphism groups have orbit accumulation points in their boundaries. ✠

First, such a construction is quite awkward: should it be "groups" or "group"? More importantly, do all the domains share the same automorphism group, or does each have its own? Does each domain have several orbit accumulation points, or just one? Clearer is the sentence

> A domain with noncompact automorphism group has an orbit accumulation point in its boundary.

When you are putting the final polish on a manuscript, look it over for general appearance. In mathematical writing, several consecutive pages of dense prose are not appealing, nor are several consecutive pages of tedious calculation. For ease of reading, the two types of mathematical writing should be interwoven. It requires only a small extra effort to produce a paper or book with comfortable stopping places on every page. The reader needs to take frequent breathers, to survey what he/she has read, to pause and look back. Make it easy for him/her to do so.

While you are thinking about the counterpoint between prose and formulas, think also about the use of displayed math versus in-text math [in TEX (see Section 6.5), the former is set off by double dollar signs `$$` while the latter is set off with single dollar signs `$`]. Long formulas are usually better displayed, for they are difficult to read when put in text. Of course *important* formulas should be displayed no matter what their length—and provided with numbers or labels if they will be mentioned later. Do not display every single formula, for that will make your paper a cumbersome read. Also do not put every formula in text, as that will make your writing tedious. A little thought will help you to strike a balance, and to use the two formats to good effect.

And now a coda on the role of English in mathematical writing. More and more, English is becoming the language of choice in mathematics. Therefore those of us who are native speakers set the standard for those who are not. We should exercise a bit of care. I have a good friend, also an excellent mathematician, who is widely admired; his fans like to emulate him. He is fond of saying (informally) "What you need here is to cook up a function f such that ...." Mathematicians of foreign extraction, who have been hearing him make this statement for years, have now developed the habit of saying "Take a function f. Now cook it for a while ...." It is a bit like having your children emulate (poorly) all your bad habits. A word to the wise should suffice.