1. Short answer

(a) Give an example of a series $\sum_{n=1}^{\infty} a_n$ whose terms approach 0, but which diverges.

The canonical example is $\sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series.

(b) Explain what it means for a series to be conditionally convergent.

A series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if it converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges.

(c) Explain the similarities in the Ratio Test and the Root Test.

Ratio Test $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$ 
If $L < 1$, the series converges
$L = 1$, we cannot tell
$L > 1$, the series diverges

Root test $\lim_{n \to \infty} \sqrt[n]{|a_n|}$ 
If $L < 1$, the series converges
$L > 1$, the series diverges
$L = 1$, we cannot tell

Both provide the same conditions as $L \in [0, \infty)$ for convergence.
2. For each series, tell whether it is convergent or divergent. For those convergent alternating series, determine whether the series absolutely converges or conditionally converges. Be sure to justify your answers by using the appropriate tests.

(a) \[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{\sqrt{n^5 + 1}} \]

Alternating Series \[ \lim_{n \to \infty} b_n = 0 \] (compare exponents)

\[ b_{n+1} > b_n \] so converges

\[ \text{Compare } \leq b_n \sim \sum \frac{1}{n^{5/2}} \]

\[ \lim_{n \to \infty} \frac{n^{5/2}}{n^{5/2}} = 1 \]

Subconvergence because the same.

(b) \[ \sum_{n=1}^{\infty} \frac{\cos(n)}{2^n} \]

\[ \sum_{n=1}^{\infty} \frac{\cos(n)}{2^n} < \sum_{n=1}^{\infty} \frac{1}{2^n} \text{ converges, geometric} \]

\[ \Rightarrow \sum_{n=1}^{\infty} \frac{\cos(n)}{2^n} \text{ is convergent.} \]

(c) \[ \sum_{n=1}^{\infty} \frac{4^n}{3^{2n}} \]

Ratio Test \[ a_n = \frac{4^n}{3^{2n}} \quad a_{n+1} = \frac{4^{n+1}}{3^{2(n+1)}} \]

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{4^{n+1}}{3^{2(n+1)}} \cdot \frac{3^{2n}}{4^n} \right| = \frac{4}{9} \leq 1 \]

So convergent
(d) \[ \sum_{n=1}^{\infty} \frac{\ln(n)}{n^3} \]

\[ \sum_{n=1}^{\infty} \frac{\ln(n)}{n^3} < \sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2} \]

\[ \boxed{C} \quad \text{Convergent (p series)} \]

(e) \[ \sum_{n=1}^{\infty} (-1)^{n+1} \sin\left(\frac{1}{n}\right) \]

\[ b_n = \sin\left(\frac{1}{n}\right) \quad \lim_{n \to \infty} b_n = 0 \]

\[ b_n > b_{n+1} \rightarrow \boxed{C} \]

Compare \[ \sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right) \text{ with } \sum_{n=1}^{\infty} \frac{1}{n} \]

\[ \lim_{n \to \infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1 \quad \text{so both Diverge} \]

\[ \Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} \sin\left(\frac{1}{n}\right) \text{ is conditionally convergent} \]
Ratio test

\[ a_n = \frac{n! (n+1)!}{(2n)!}, \quad a_{n+1} = \frac{(n+1)! (n+2)!}{(2(n+1))!} \]

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(n+1)! (n+2)!}{n! (2n)!} \cdot \frac{2n}{n+2} \]

\[ = \lim_{n \to \infty} \frac{(n+2)(n+1)}{(2n+2)(2n+1)} = \frac{1}{4} < 1 \text{ so } \text{Convergent} \]

3. (Bonus): Alice and Bob decide to play the game again. (Take turns tossing a fair coin, first head wins, Alice goes first.). This time, to make things fairer, Bob will get 2 tosses on each turn where Alice gets only one. Is the game now fair? Explain.

Alice still has an advantage. She will win 50% of the time on her first toss, so more than 50% of the time overall.

\[ P(A \text{ wins}) = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \cdots \]

\[ = \frac{1}{2} + \frac{1}{4} + \frac{1}{16} + \cdots = \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{8}} = \frac{8}{7} = \frac{4}{7} \]