**Problem:** For each positive integer $n$, the formula

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n + 2) = \frac{n(n + 1)(2n + 7)}{6}$$

is valid.

**Proof:** (formal style; it is good to do a few proofs this way) We will use the Principle of Mathematical Induction. Let $S$ be the set of all positive integers $n$ such that

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n + 2) = \frac{n(n + 1)(2n + 7)}{6}.$$

Since $1 \cdot 3 = (1 \cdot 2 \cdot 9)/6$, it is clear that $1 \in S$. Suppose that $k \in S$ for some positive integer $k$. We then have

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + (k + 1)(k + 3) \quad (\text{substituting } k + 1 \text{ for } n)$$

$$= 1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + k(k + 2) + (k + 1)(k + 3) \quad (\text{include extra term})$$

$$= \frac{k(k + 1)(2k + 7)}{6} + (k + 1)(k + 3) \quad (\text{since } k \in S)$$

$$= \frac{k + 1}{6} (2k^2 + 7k + 6k + 18) \quad (\text{factoring})$$

$$= \frac{k + 1}{6} (k + 2)(2k + 9) \quad (\text{more factoring})$$

$$= \frac{(k + 1)(k + 2)(2k + 9)}{6}. \quad (\text{the form we want})$$

This shows that $k + 1 \in S$. By the Principle of Mathematical Induction, it follows that $S = \mathbb{Z}^+$. Hence,

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n + 2) = \frac{n(n + 1)(2n + 7)}{6}$$

for all positive integers $n$.

**Proof:** (informal style; more common in textbooks) The formula given in the statement of the problem is clearly true for $n = 1$. Suppose that the formula is valid for some positive integer $k$. Then

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + (k + 1)(k + 3) = \frac{k(k + 1)(2k + 7)}{6} + (k + 1)(k + 3)$$

$$= \frac{k + 1}{6} (2k^2 + 7k + 6k + 18)$$

$$= \frac{(k + 1)(k + 2)(2k + 9)}{6},$$

showing that the formula is valid for $k + 1$ as well. The result now follows by the Principle of Mathematical Induction.
Here are three PMI proofs of this same result, each with one or more errors; be certain you can spot the errors.

**Proof:** We will use the Principle of Mathematical Induction. Since $1 \cdot 3 = (1 \cdot 2 \cdot 9)/6$, it is clear that the formula works when $n = 1$. Suppose that $k \in S$ for some positive integer $k$. We then have

\[
1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + (k+1)(k+3) = \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3)
\]

\[
= \frac{k+1}{6}\left(2k^2 + 7k + 6k + 18\right)
\]

\[
= \frac{(k+1)(k+2)(2k+9)}{6},
\]

so $k+1 \in S$. By the Principle of Mathematical Induction,

\[
1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}
\]

for all positive integers $n$.

**Proof:** We will use the Principle of Mathematical Induction. Since $1 \cdot 3 = (1 \cdot 2 \cdot 9)/6$, it is clear that the formula works when $n = 1$. Now suppose that the formula is valid for every positive integer $k$. Then

\[
1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + (k+1)(k+3) = \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3)
\]

\[
= \frac{k+1}{6}\left(2k^2 + 7k + 6k + 18\right)
\]

\[
= \frac{(k+1)(k+2)(2k+9)}{6},
\]

so the formula works for all $n$.

**Proof:** We will use the Principle of Mathematical Induction. Let $S$ be the set of all positive integers $n$ such that

\[
1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}.
\]

Since $1 \cdot 3 = (1 \cdot 2 \cdot 9)/6$, it is clear that $1 \in S$. Suppose that $k \in S$ for some positive integer $k$. We then have

\[
1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + (k+1)(k+3) = \frac{(k+1)(k+2)(2k+1+7)}{6}
\]

\[
= \frac{k+1}{6}\left(2k^2 + 7k + 6k + 18\right) = \frac{k+1}{6}\left(2k^2 + 13k + 18\right).
\]

This shows that $k+1 \in S$. By the Principle of Mathematical Induction, it follows that $S \in Z^+$. Hence,

\[
1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}
\]

for all positive integers $n$. 

\[\blacksquare\]
Here are three correct proofs for a different result; study them carefully.

**Theorem:** For each positive integer \( n \), the integer \( 3^{2n+1} + 2^n + 2 \) is divisible by 7.

**Proof 1:** We will use the Principle of Mathematical Induction. Let \( S \) be the set of all positive integers \( n \) for which \( 3^{2n+1} + 2^n + 2 \) is divisible by 7. When \( n = 1 \), we see that \( 3^3 + 2^3 = 35 \) is divisible by 7. This shows that \( 1 \in S \). Now suppose that \( k \in S \) for some positive integer \( k \). Since \( 3^{2k+1} + 2^k + 2 \) is divisible by 7, there exists an integer \( q \) such that \( 3^{2k+1} + 2(3^{2k+1} + 2^k + 2) = 7q \). We then have (using one of several options)

\[
3^{2k+3} + 2^{k+3} = 3^2 \cdot 3^{2k+1} + 2 \cdot 2^{k+2} \\
= 7 \cdot 3^{2k+1} + 2(3^{2k+1} + 2^k + 2) \\
= 7 \cdot 3^{2k+1} + 2(7q) \\
= 7(3^{2k+1} + 2q),
\]

revealing that 7 divides \( 3^{2k+3} + 2^{k+3} \). This means that \( k + 1 \in S \). By the Principle of Mathematical Induction, \( S = \mathbb{Z}^+ \). Hence, the integer \( 3^{2n+1} + 2^n + 2 \) is divisible by 7 for each positive integer \( n \).  

**Proof 2:** We will use the Principle of Mathematical Induction. For each positive integer \( n \), let \( P_n \) be the statement that \( 3^{2n+1} + 2^n + 2 \) is divisible by 7. Since \( 3^3 + 2^3 = 35 \) is divisible by 7, it is clear that \( P_1 \) is true. Suppose that \( P_k \) is true for some positive integer \( k \). Since \( 3^{2k+1} + 2^k + 2 \) is divisible by 7, there exists an integer \( q \) such that \( 3^{2k+1} + 2^k + 2 = 7q \). We then have (using one of several options)

\[
3^{2k+3} + 2^{k+3} = 3^2 \cdot 3^{2k+1} + 2 \cdot 2^{k+2} \\
= 9(7q - 2^{k+2}) + 2 \cdot 2^{k+2} \\
= 63q - 7 \cdot 2^{k+2} \\
= 7(9q - 2^{k+2}),
\]

revealing that 7 divides \( 3^{2k+3} + 2^{k+3} \). This means that \( P_{k+1} \) is true. By the Principle of Mathematical Induction, all of the \( P_n \) statements are true, that is, the integer \( 3^{2n+1} + 2^n + 2 \) is divisible by 7 for each positive integer \( n \).  

**Proof 3:** The statement is easily seen to be true when \( n = 1 \). Suppose that \( 3^{2k+1} + 2^k + 2 \) is divisible by 7 for some positive integer \( k \) and choose an integer \( q \) such that \( 3^{2k+1} + 2^k + 2 = 7q \). We then have

\[
3^{2k+3} + 2^{k+3} = 3^2 \cdot 3^{2k+1} + 2 \cdot 2^{k+2} \\
= 9(3^{2k+1} + 2^k + 2) - 7 \cdot 2^{k+2} \\
= 9(7q) - 7 \cdot 2^{k+2} \\
= 7(9q - 2^{k+2}),
\]

revealing that 7 divides \( 3^{2k+3} + 2^{k+3} \). By the Principle of Mathematical Induction, the integer \( 3^{2n+1} + 2^n + 2 \) is divisible by 7 for each positive integer \( n \).