The derivative of a function \( f \) is another function \( f' \) defined by

\[
f'(x) = \lim_{v \to x} \frac{f(v) - f(x)}{v - x}
\]

or (equivalently)

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

for each value of \( x \) in the domain of \( f \) for which the limit exists.

The number \( f'(c) \) represents the slope of the graph \( y = f(x) \) at the point \((c, f(c))\). It also represents the rate of change of \( y \) with respect to \( x \) when \( x \) is near \( c \).

An equation for the tangent line to the curve \( y = f(x) \) when \( x = c \) is

\[y - f(c) = f'(c)(x - c)\]

Using the definition of the derivative, it is possible to establish the following derivative formulas.

\[
\begin{align*}
\frac{d}{dx} x^r &= rx^{r-1}, \quad r \neq 0 \\
\frac{d}{dx} \sin x &= \cos x \\
\frac{d}{dx} \cos x &= -\sin x \\
\frac{d}{dx} \arcsin x &= \frac{1}{\sqrt{1 - x^2}} \\
\frac{d}{dx} \arccos x &= -\frac{1}{\sqrt{1 - x^2}} \\
\frac{d}{dx} e^x &= e^x \\
\frac{d}{dx} \arctan x &= \frac{1}{1 + x^2} \\
\frac{d}{dx} \arccot x &= -\frac{1}{1 + x^2} \\
\frac{d}{dx} \log_a |x| &= \frac{1}{(\ln a)x}, \quad a > 0 \\
\frac{d}{dx} a^x &= (\ln a)a^x, \quad a > 0 \\
\frac{d}{dx} \sec x &= \sec x \tan x \\
\frac{d}{dx} \csc x &= -\csc x \cot x \\
\frac{d}{dx} \arccsc x &= -\frac{1}{|x|\sqrt{x^2 - 1}} \\
\frac{d}{dx} \arccot x &= \frac{1}{|x|\sqrt{x^2 - 1}} \\
\frac{d}{dx} \arccsc x &= -\frac{1}{|x|\sqrt{x^2 - 1}}
\end{align*}
\]

**Product Rule:**

\[
\frac{d}{dx} (F(x)G(x)) = F(x)G'(x) + G(x)F'(x)
\]

**Quotient Rule:**

\[
\frac{d}{dx} \left( \frac{F(x)}{G(x)} \right) = \frac{G(x)F'(x) - F(x)G'(x)}{(G(x))^2}
\]

**Chain Rule:**

\[
\frac{d}{dx} F(G(x)) = F'(G(x)) G'(x)
\]

**Mean Value Theorem:** If \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\), then there exists a number \( c \) in \((a, b)\) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

This theorem states that there is a point \( c \) for which the instantaneous rate of change of \( f \) at \( c \) \((f'(c))\) equals the average (mean) rate of change of \( f \) on \([a, b]\) \(((f(b) - f(a))/(b - a))\). You should be familiar with a graphical interpretation of this theorem; it involves two parallel lines.

The Mean Value Theorem can be used to prove the following three facts.
1. If \( f' \) is positive (negative) on an interval \( I \), then \( f \) is increasing (decreasing) on \( I \). This fact makes it possible to use \( f' \) to determine the values of \( x \) for which \( f \) has a relative maximum value or a relative minimum value. The first step is to find the critical points of \( f \): points \( x \) in the domain of \( f \) for which either \( f'(x) = 0 \) or \( f'(x) \) does not exist. Then the First Derivative Test (or perhaps the Second Derivative Test) can be used to determine the nature of the critical point.

2. If \( f'' \) is positive (negative) on an interval \( I \), then \( f \) is concave up (concave down) on \( I \). An inflection point occurs where the graph changes concavity. Possible inflection points occur when \( f''(x) = 0 \), but it is necessary to check that the concavity actually changes at such points.

3. If \( f' = g' \) on an interval \( I \), then there is a constant \( C \) such that \( g(x) = f(x) + C \) for all \( x \) in \( I \).

A function \( f \) is continuous at a number \( c \) if \( \lim_{x \to c} f(x) = f(c) \). This fact guarantees that the graph of \( f \) does not have a break at \( c \). An important theorem states: If \( f \) is differentiable at \( c \), then \( f \) is continuous at \( c \). However, the converse is false; the function \( f(x) = |x| \) is continuous at 0 but not differentiable at 0.

**Intermediate Value Theorem:** If \( f \) is continuous on a closed interval \([a, b]\) and \( v \) is any number between \( f(a) \) and \( f(b) \), then there is a number \( c \) in \((a, b)\) such that \( f(c) = v \).

**Extreme Value Theorem:** If \( f \) is continuous on a closed interval \([a, b]\), then there exist numbers \( c \) and \( d \) in \([a, b]\) such that \( f(c) \leq f(x) \leq f(d) \) for all \( x \) in \([a, b]\). (The number \( f(c) \) is the minimum value of \( f \) on \([a, b]\) and the number \( f(d) \) is the maximum value of \( f \) on \([a, b]\).

**Formal definition of limit:** Let \( f \) be defined on some open interval containing the point \( c \), except possibly at \( c \). Then \( \lim_{x \to c} f(x) = L \) if for each \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( |f(x) - L| < \epsilon \) for all \( x \) that satisfy \( 0 < |x - c| < \delta \).

A function \( f \) has a vertical asymptote \( x = c \) if either \( \lim_{x \to c^+} |f(x)| = \infty \) or \( \lim_{x \to c^-} |f(x)| = \infty \).
A function \( f \) has a horizontal asymptote \( y = d \) if either \( \lim_{x \to \infty} f(x) = d \) or \( \lim_{x \to -\infty} f(x) = d \).

Various algebraic techniques (factoring, expanding, finding a common denominator, multiplying by the conjugate) can be used to evaluate limits. The following rule is sometimes useful for computing limits of the form \( 0/0 \) or \( \infty/\infty \); these are known as indeterminate forms. The suitable conditions mentioned in the hypotheses involve continuity and differentiability conditions that will always be met by the functions we encounter. (For the record, other indeterminate forms include \( 0 \cdot \infty, \infty - \infty, 1^\infty, \) and \( \infty^0 \).)

**L’Hôpital’s Rule:** Under suitable conditions on the functions \( f \) and \( g \), if either \( \lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x) \) or \( \lim_{x \to a} f(x) = \infty = \lim_{x \to a} g(x) \), then \( \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \), assuming that the latter limit exists. (The limits here can be of any type; \( x \to a, x \to a^+, x \to a^-, x \to \infty, x \to -\infty \).)