Calc I Review:

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- If \( f \) is differentiable at \( x = a \), then it is locally linear. The linearization \( L \) is an approximation to \( f \) at \( x = a \),

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**How should “Differentiable” be defined?**

A function \( z = f(x, y) \) should be “differentiable” at \( (a, b) \) if it is locally linear there.
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How should “Differentiable” be defined?

A function $z = f(x, y)$ should be “differentiable” at $(a, b)$ if it is locally linear there.

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

$$f(x, y) \approx L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$
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This *should* guarantee the existence of the partial derivatives and the continuity of $z = f(x, y)$ at a point $(a, b)$. 
Main Point:

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“Differentiable” is a stronger condition than existence of the partial derivatives,

But if the partial derivatives are continuous at $(a, b)$, then $f$ is differentiable there (in the sense of being locally linear).
Example:

\[ f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases} \]
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Show that \(f_x(0, 0)\) and \(f_y(0, 0)\) both exist:

\[
f_x(0, 0) = \lim_{h \to 0} \frac{f(0 + h, 0) - f(0, 0)}{h} = \]

This function is NOT continuous at the origin (consider \(y = x\) and \(y = -x\)). The partial derivatives may exist, even though the function is not continuous at a point.
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Summary:
The **definition** of differentiability is somewhat complicated. We will use the following theorem instead:
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Differentiability Theorem
If the partial derivatives exist and are continuous on a small disk centered at \((a, b)\), then \(z = f(x, y)\) is differentiable at \((a, b)\).
If \( z = f(x, y) \) is differentiable at \((a, b)\), then we can use the tangent plane to approximate it. That is, either directly:

\[
z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)
\]

Or indirectly: Let \( dx = \Delta x = x - a \) and \( dy = \Delta y = y - b \). Then the total differential \( dz \) is approximately \( \Delta z \),

\[
\Delta z \approx dz = f_x(a, b) \, dx + f_y(a, b) \, dy
\]
Find the linear approximation to $f(x, y) = \ln(x - 3y)$ at (7, 2) and use it to approximate $f(6.9, 2.06)$

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SOLUTION:

\[
f_x(7, 2) = \left. \frac{1}{x - 3y} \right|_{x=7,y=2} = 1 \quad f_y(7, 2) = \left. \frac{-3}{x - 3y} \right|_{x=7,y=2} = -3
\]

Therefore, using $f(7, 2) = \ln(1) = 0$, we have:

\[
f(x, y) \approx \]

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**SOLUTION:**

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\begin{align*}
\frac{\partial f}{\partial x}(7, 2) &= \frac{1}{x - 3y} \bigg|_{x=7, y=2} = 1 \\
\frac{\partial f}{\partial y}(7, 2) &= \frac{-3}{x - 3y} \bigg|_{x=7, y=2} = -3
\end{align*}
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Therefore, using \( f(7, 2) = \ln(1) = 0 \), we have:

\[
f(x, y) \approx 0 + 1(x - 7) - 3(y - 2) = x - 3y - 1
\]

Now, use \( dx = \Delta x = 6.9 - 7.0 = -0.1 \) and \( dy = \Delta y = 2.06 - 2 = 0.06 \)
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\[
f(6.9, 2.06) \approx 0 + 1 \cdot (-0.1) - 3(0.06) = -0.28
\]