Definition

A function $z = f(x, y)$ has a local minimum at a point $(a, b)$ if there is a disk about $(a, b)$ so that $f(a, b) \leq f(x, y)$ for all $(x, y)$ in the disk.

Vocab: The point $(a, b)$ is the minimizer, the value $f(a, b)$ is the minimum.
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Definition
A function \( z = f(x, y) \) has a global min (max) at a point \((a, b)\) in a given region \(D\) if \(f(a, b)\) is the smallest (largest) point in all of \(D\) (could be equality, too- There could be multiple max's and min's).
As in Calc I, we have the Extreme Value Theorem:

**Theorem**

*If \( z = f(x, y) \) is continuous on a closed and bounded region in the plane, \( D \), then \( f \) attains a global max and min on \( D \).*
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**Definition**

The critical points of \( z = f(x, y) \) are points where \( \nabla f = 0 \) or either (or both) partial derivatives do not exist.
In the case that the EVT applies (global max/min on a closed and bounded domain), the candidates for where the max/min can occur:
In the case that the EVT applies (global max/min on a closed and bounded domain), the candidates for where the max/min can occur:

- Critical points
- Boundary

Check them, and find the max/min on each (build a table).
Example: Find the global max and global min:

\[ f(x, y) = 5 + x^2 + x - 2y^2 \quad -1 \leq x \leq 1, \quad -1 \leq y \leq 1 \]
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SOLUTION: Find critical points:

\[ f_x(x, y) = 2x + 1 \quad f_y(x, y) = -4y \quad \Rightarrow \quad (-1/2, 0) \]

Value of \( f \) at the critical point: 4.75.
Check the boundary:

- \( f(x, y) = 5 + x^2 + x - 2y^2 \) for \( x = 1, -1 \leq y \leq 1 \):
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\[
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\[
\begin{array}{c|c}
  y & f(1, y) \\
  \hline
  -1 & f(1, -1) = 5 \\
  0 & f(1, 0) = 7 \\
  1 & f(1, 1) = 5 \\
\end{array}
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For \(y = -1\), we have the same function and same interval.
Conclusion:
The global maximum is 7, it occurs at \((1, 0)\) on the boundary. The global minimum is 2.75, it occurs twice on the boundary, at \((-1/2, \pm 1)\).
Local Extrema

To find local extrema, in Calc I we had the first and second derivative tests. It is not easy to find a substitute- A surface can be both CU and CD at a saddle point.
The Second Derivatives Test

Let

\[
D(a, b) = \left| \begin{array}{cc}
 f_{xx}(a, b) & f_{xy}(a, b) \\
 f_{yx}(a, b) & f_{yy}(a, b)
\end{array} \right|
\]

Then, if

\[ D > 0 \text{ and } f_{xx}(a, b) > 0 \] (takes the place of CU), 
\[ f(a, b) \] is a local min.

\[ D > 0 \text{ and } f_{xx}(a, b) < 0 \] (takes the place of CD), 
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\[ D(a, b) = \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{vmatrix} = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b) \]
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\[ f(x, y) = 3y^3 + 9y^2 - 3xy + \frac{1}{2}x^2 + 9y - 9x \]
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And the second partials:

\[ f_{xx} = 1 \quad f_{xy} = -3 \quad f_{yy} = 18y + 18 \]
Critical points

\[-3y + x - 9 = 0 \quad \text{and} \quad 9y^2 + 18y - 3x + 9 = 0\]

Substitute \(x = 3(y + 3)\) into the second to eliminate \(x\):

\[9y^2 + 18y - 9(y + 3) + 9 = 9y^2 + 9y - 18 = 0\]

\[y^2 + y - 2 = 0\]

Therefore, \(y = -2\) and \(y = 1\). Backsub to get the ordered pairs: \((3, -2), (12, 1)\)
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\[(3, -2) \quad (12, 1)\]
Do the Second Derivatives test on each CP; simplify first:

\[ D(x, y) = (1)(18y + 18) - (-3)^2 = 18y + 9 \]
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So, at \((3, -2)\), \(D(3, -2) = -36 + 9 < 0\) so that is a SADDLE POINT.
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At (12, 1), we have \( D(12, 1) = 18 + 9 > 0 \), and \( f_{xx} > 0 \), so we have a LOCAL MIN.
Example

Find the local maximum, minimum and saddle points. Verify your answer by locating these points on the plot of level curves.

\[ g(x, y) = xy(1 - x - y) \]
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SOLUTION: Find the critical points, then classify according to the Second Derivatives Test. First, we’ll compute the partial derivatives (and the seconds):

\[ g_x = y(1 - 2x - y) \quad g_{xx} = -2y \quad g_{xy} = 1 - 2x - 2y \]
\[ g_y = x(1 - x - 2y) \quad g_{yy} = -2x \]
Solving for the critical points,

\[ y(1 - 2x - y) = 0 \implies y = 0 \text{ or } y = 1 - 2x \]
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So far, we have two critical points, \((0, 0)\) and \((1, 0)\). If \( y = 1 - 2x \), then:

\[ x(1 - x - 2(1 - 2x)) = 0 \quad \Rightarrow \quad x = 0 \quad \text{or} \quad x = 1/3 \]

Now we have two more fixed points: \((0, 1)\) or \((1/3, 1/3)\).
In each case apply the Second Derivatives Test:

\[ D = g_{xx} g_{yy} - g_{xy}^2 = 4xy - (1 - 2x - 2y)^2 \]
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<td></td>
</tr>
</tbody>
</table>
In each case apply the Second Derivatives Test:

\[ D = g_{xx}g_{yy} - g_{xy}^2 = 4xy - (1 - 2x - 2y)^2 \]

<table>
<thead>
<tr>
<th>Point</th>
<th>( D )</th>
<th>( g_{xx} )</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>-1</td>
<td>N/A</td>
<td>Saddle</td>
</tr>
<tr>
<td>(1, 0)</td>
<td>-1</td>
<td>N/A</td>
<td>Saddle</td>
</tr>
<tr>
<td>(0, 1)</td>
<td>-1</td>
<td>N/A</td>
<td>Saddle</td>
</tr>
<tr>
<td>(1/3, 1/3)</td>
<td>1/3</td>
<td>-2/3</td>
<td>Local Max</td>
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Here is the contour plot, and we see the saddles and local max:
Example:

Find the local max, min and saddle points:

\[ f(x, y) = x^2 y e^{-x^2 - y^2} \]

SOLUTION: First compute critical points:
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\[ f(x, y) = x^2 ye^{-x^2-y^2} \]

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\[ f_x(x, y) = 2xy (1 - x^2) e^{-x^2-y^2} \quad f_y(x, y) = x^2 (1 - 2y^2) e^{-x^2-y^2} \]

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and second derivatives:

\[ f_{xx} = (2y - 10x^2 y + 4x^4 y)e^{-x^2 - y^2} \quad f_{yy} = (4x^2 y^3 - 6x^2 y)e^{-x^2 - y^2} \]

and

\[ f_{xy} = 2x(1 - x^2 - 2y^2 + 2x^2 y^2)e^{-x^2 - y^2} \]
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From the graph, we see that if $y > 0$, then points $(0, y)$ are where local minima occur, and if $y > 0$, then $(0, y)$ are where local maxima occur. These would be difficult to determine without the graph.
If $D = 0$, some complicated behaviors can occur. In this example, we have

$$f(x, y) = x^3 - 3xy^2$$

Below is the surface, called a “Monkey Saddle”, and the corresponding contour plot.