1. Short Answer:

(a) What is the definition of the directional derivative?

SOLUTION: Given $z = f(x, y)$, the directional derivative at $(a, b)$ in the direction of the unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$ is:

$$D_u f(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$$

(b) Find the differential: $w = xye^{xz}$

SOLUTION: If $f(x, y, z) = xye^{xz}$, then:

$$dw = f_x \, dx + f_y \, dy + f_z \, dz = (ye^{xz} + xye^{xz}) \, dx + xze^{xz} \, dy + x^2 y e^{xz} \, dz$$

(c) If $z = x^2y + 3xy^4$ where $x = \sin(2t)$ and $y = \cos(2t)$, find $dz/dt$ when $t = 0$.

SOLUTION:

$$\frac{dz}{dt} = z_x \frac{dx}{dt} + z_y \frac{dy}{dt}$$

We note that at $t = 0$, then $x = 0$ and $y = 1$:

$$z_x = 2xy + 3y^4 \Rightarrow z_x(0, 1) = 3 \quad \frac{dx}{dt} = 2 \cos(0) = 2$$

Similarly,

$$z_y = x^2 + 12xy^3 \Rightarrow z_y(0, 1) = 0 \quad \frac{dy}{dt} = -2 \sin(0) = 0$$

Therefore, $dz/dt = 6$

2. Consider the function

$$f(x, y) = \begin{cases} \frac{2x^2 + 3y^2}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Compute $f_x(0, 0)$ by using the definition of the partial derivative.

SOLUTION:

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{1}{h} \cdot \frac{2h^2}{h} = 2$$

3. Let $f(x, y) = 1 + 2x\sqrt{y}$. Find the rate of change of $f$ at $(3, 4)$ in the direction of the vector $\langle 4, -3 \rangle$.

SOLUTION: The gradient is $\langle 2\sqrt{y}, xy^{-1/2} \rangle$

$$D_u f(3, 4) = \nabla f(3, 4) \cdot \mathbf{u} = \langle 4/3, 2 \rangle \cdot \frac{1}{5} \langle 4, -3 \rangle = \frac{23}{10}$$

4. Limit

(a) Show that this limit exists at the origin by using the Squeeze Theorem:

$$\lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^2}$$

SOLUTION:

$$0 \leq \frac{|x|y^2}{x^2 + y^2} \leq \frac{|x|y^2}{y^2} = |x|$$

Take the limit as $x \to 0$ and they all are zero (which forces the second expression to be zero).
(b) Show that this limit does not exist:

\[
\lim_{{(x,y)\to (0,0)}} \frac{xy + y^3}{x^2 + y^2}
\]

(NOTE: On the exam, I would give you the figure. Since this is a take-home problem, you can plot it using Wolfram Alpha to help you).

SOLUTION: Along \( y = x \), we have:

\[
\frac{xy + y^3}{x^2 + y^2} = \frac{x^2 + x^3}{2x^2} = \frac{x^2(1 + x)}{2x^2} = \frac{1 + x}{2}
\]

Therefore, the limit as \( x \to 0 \) along \( y = x \) is \( \frac{1}{2} \). If we go to zero along the \( x \) axis (\( y = 0 \)), we get:

\[
\frac{0 + 0}{x^2 + 0} = 0
\]

so the limit is zero. Since the limits are different, the overall limit does not exist.

5. Let \( f(x, y) = x^2 + y^2 - 2x - 4y \) be our height at \((x, y)\).

(a) If we are at \((1, 1)\), in which direction should we move in order to move uphill the fastest? What is the rate of change if we move in that direction?

SOLUTION: Move in the direction of \( \nabla f(1, 1) \), which is:

\[
\langle 2x - 2, 2y - 4 \rangle|_{{x=1, y=1}} = \langle 0, -2 \rangle
\]

If we move in that direction, then the directional derivative is 2.

(b) At \((1, 1)\), the gradient of \( f \) should be orthogonal to what level curve (and what kind of a curve is it?)

SOLUTION: The level curve is at \( 1^2 + 1^2 - 2 - 4 = -4 \), or

\[
x^2 + y^2 - 2x - 4y = -4
\]

(c) Find the equation of the tangent line and the normal line to the level curve at \((1, 1)\).

SOLUTION: The normal line goes through \((1, 1)\) in the direction of \( <0, -2> \), so in parametric form: \( x = 1, y = 1 - 2t \). Note that this is the equation of a vertical line- So the tangent line is the horizontal line going through \((1, 1)\), which is \( y = 1 \).

You could also find the tangent line using the gradient:

\[
0(x - 1) - 2(y - 1) = 0 \quad \Rightarrow \quad y = 1
\]

Or you could find the slope of the tangent line using Calc I:

\[
2x + 2yy' - 2 - 4y' = 0 \quad \Rightarrow \quad y' = 0
\]

so \( y - 1 = 0(x - 1) \) or \( y = 1 \).

6. Suppose we have computed the directional derivative at \((1, 2)\) in the direction of the unit vector \( \langle u_1, u_2 \rangle \), and it was:

\[
D_u f(1, 2) = u_1 + u_2^2
\]

(a) Compute \( f_x(1, 2) \) and \( f_y(1, 2) \):

SOLUTION: The idea is to recall that the directional derivative is more general than the partial derivatives. We can compute \( f_x(1, 2) \) by taking \( u = \langle 1, 0 \rangle \) and \( f_y(1, 2) \) by taking \( u = \langle 0, 1 \rangle \). In that case, we get:

\[
f_x(1, 2) = 1 \quad f_y(1, 2) = 1
\]
(b) Is \( f \) differentiable at \((1,1)\)? (Be specific)

**SOLUTION:** **Typo:** The point \((1,1)\) should be \((1,2)\), since we have no information about \((1,1)\). At the point \((1,2)\) the function is NOT differentiable. If it were, the directional derivative would be:

\[
D_uf(1,2) = \nabla f(1,2) \cdot \langle u_1, u_2 \rangle = u_1 + u_2
\]

However, we said that the directional derivative is \(u_1 + u_2^2\).

**NOTE:** The idea we were testing: Is it always true that \(D_uf(a,b) = \nabla f(a,b) \cdot \vec{u}\)? No, except if \(f\) is differentiable at \((a,b)\).

7. If \( \vec{r}(t) = \langle \sin(t), \cos(t), t \rangle \)
   
   (a) Find \( \int_0^7 \vec{r}(t) \, dt = \langle -\cos(t), \sin(t), t^2/2 \rangle \big|_0^7 = \langle 2, 0, \pi/2 \rangle \)
   
   (b) Find \( \mathbf{T}, \mathbf{N} \).

**SOLUTION:**

\[
\mathbf{T} = \frac{\vec{r}'}{|\vec{r}'|} = \frac{1}{\sqrt{2}} \langle \cos(t), -\sin(t), 1 \rangle
\]

The normal vector \( \mathbf{N}(t) \) is (unit) derivative of \( \mathbf{T} \),

\[
\mathbf{N}(t) = \langle -\sin(t), -\cos(t), 0 \rangle
\]

(c) Find the arc length function \( s(t) \).

**SOLUTION:**

\[
s(t) = \int_0^t |\vec{r}'(u)| \, du = \int_0^t \sqrt{\cos^2(u) + \sin^2(u) + 1} \, du = \sqrt{2}t
\]

8. Find the maximum and minimum of \( f(x, y) = x^2y \) on the domain where \( x^2 + y^2 \leq 3 \).

**SOLUTION:** This is a closed and bounded domain, and \( f \) is a continuous function, so the global maximum and global minimum will occur either at a critical point or on the boundary. First the critical points:

\[
\nabla f = \langle 2xy, x^2 \rangle
\]

The critical points are where \( x = 0 \) (\( y \) can be anything). In these cases, the value of \( f \) is zero.

On the boundary, \( x^2 = 3 - y^2 \), so \( f \) can be written as a function of one variable,

\[
f(y) = (3 - y^2)y = 3y - y^3 \quad \quad -\sqrt{3} \leq y \leq \sqrt{3}
\]

On the endpoints, \( y = \pm \sqrt{3} \), \( f(y) = 0 \), so the only points that remain are where \( f'(y) = 0 \):

\[
f'(y) = 3 - 3y^2 = 0 \quad \Rightarrow \quad y = \pm 1
\]

If \( y = 1, x = \pm \sqrt{2} \), and if \( y = -1, x = \pm \sqrt{2} \). At each of the first two points, \( f(x,y) = 2 \) and at each of the last two points, \( f(x,y) = -2 \), so these are the global max and min.

**Extra:** We could also use Lagrange Multipliers (14.8) to solve this.

9. Find and classify the critical points using the second derivatives test:

\[
f(x, y) = x^3 - 6xy + 8y^3
\]

Find the critical points:

\[
3x^2 - 6y = 0 \quad \text{and} \quad -6x + 24y^2 = 0
\]

Putting these together, we find that \((0,0)\) and \((1,1/2)\) are the two critical points. Test each one, noting that:

\[
D = \begin{vmatrix} 6x & -6 \\ -6 & 48y \end{vmatrix} = 288xy - 36
\]

At \((0,0)\), \( D < 0 \), so the origin is a SADDLE POINT.

At \((1,1/2)\), \( D > 0 \) and \( f_{xx} > 0 \), so that point corresponds to a LOCAL MIN.