Selected Solutions 13.3

13.3, 3. This one had some tricky algebra:

\[ \mathbf{r}'(t) = \langle \sqrt{2}, e^t, -e^{-t} \rangle \]

so that the magnitude is:

\[
|\mathbf{r}'(t)| = \sqrt{2 + e^{2t} + e^{-2t}} \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}
\]

Therefore, the arc length is:

\[
\int_0^1 |\mathbf{r}'(t)| \, dt = \left. (e^t + e^{-t}) \right|_0^1 = e - e^{-1}
\]

13.3, 11. First, find the curve of intersection between \( x^2 = 2y \) and \( 3z = xy \).

Notice that \( y = \frac{1}{2} x^2 \) and \( z = \frac{1}{3} xy \). Therefore, it might be natural to let \( x \) be the free parameter, \( t \) and using these equations, the curve of intersection is:

\[ x = t \quad y = \frac{1}{2} t^2 \quad z = \frac{1}{6} t^3 \]

Now find the length of the curve from the origin \((t = 0)\) to the point \((6, 8, 36)\) which corresponds to \( t = 6 \).

Compute the arc length:

\[
|r'(t)| = \sqrt{1 + t^2 + \frac{1}{4} t^4}
\]

We need to integrate this, so some simplification is in order- Notice that this is a perfect square:

\[
\left( \frac{1}{2} t^2 + 1 \right)^2 = \frac{1}{4} t^4 + t^2 + 1
\]

Therefore, the arc length integral becomes:

\[
\frac{1}{2} \int_0^6 t^2 + 2 \, dt = \frac{1}{2} \left[ \frac{1}{3} t^3 + 2t \right]_0^6 = 6 + 36 = 42
\]

13.3, 16. To re-parameterize with respect to arc length, we:

- Find the arc length function, \( s(t) = \int_0^t |r''(t)| \, dt \). In this case, it looks worse than it is:

\[
r''(t) = \left\langle -2(t^2 + 1)^{-2}(2t), \frac{2(t^2 + 1) - 2t(2t)}{(t^2 + 1)^2} \right\rangle = \left\langle \frac{-4t}{(t^2 + 1)^2}, \frac{-2t^2 + 2}{(t^2 + 1)^2} \right\rangle
\]

And

\[
|r''(t)|^2 = \frac{16t^2}{(t^2 + 1)^4} + \frac{(-2t^2 + 2)^2}{(t^2 + 1)^4} = \frac{16t^2 + 4t^4 - 8t^2 + 4}{(t^2 + 1)^4} = \frac{4(t^4 + 2t^2 + 1)}{(t^2 + 1)^4} = \frac{4}{(t^2 + 1)^2}
\]

Therefore, the arc length function is:

\[
s(t) = \int_0^t \frac{2}{u^2 + 1} \, du = 2 \left( \tan^{-1}(t) - \tan^{-1}(0) \right) = 2 \tan^{-1}(t)
\]
• Invert to get \( t \) in terms of \( s \):
\[
s = 2 \tan^{-1}(t) \quad \Rightarrow \quad t = \tan\left(\frac{s}{2}\right)
\]

• Now we simply substitute back into the expression for the curve \( \vec{r}(t) \), noting that the parameterization is with respect to arc length \( s \):
\[
\vec{r}(s) = \left\langle \frac{2}{\tan^2(s/2) + 1} - 1, \frac{2 \tan(s/2)}{\tan^2(s/2) + 1} \right\rangle
\]
For fun, you might try to simplify this (not necessary at this point). If you do, you end up with: \( \vec{r}(s) = (\cos(s), \sin(s)) \)

13.3, 17. This is pretty straightforward computation:
\[
\vec{r}'(t) = (2 \cos(t), 5, -2 \sin(t)) \quad \Rightarrow \quad |\vec{r}'(t)| = \sqrt{4 \cos^2(t) + 25 + 4 \sin^2(t)} = \sqrt{29}
\]
Therefore,
\[
\mathbf{T}(t) = \frac{1}{|\vec{r}'(t)|} \vec{r}'(t) = \frac{1}{\sqrt{29}} (2 \cos(t), 5, -2 \sin(t))
\]
Now compute the unit normal vector \( \mathbf{N}(t) \). Note: For easier computation, keep the constant out in front.
\[
\mathbf{T}'(t) = \frac{1}{\sqrt{29}} (-2 \sin(t), 0, -2 \cos(t)) \quad \Rightarrow \quad |\mathbf{T}'(t)| = \frac{2}{\sqrt{29}}
\]
Therefore,
\[
\mathbf{N}(t) = \frac{1}{2} ( -2 \sin(t), 0, -2 \cos(t))
\]

13.3, 19. The algebra trick from Exercise 3 shows up again here:
\[
\vec{r}'(t) = (\sqrt{2}, e^t, -e^{-t})
\]
And the magnitude:
\[
|\vec{r}'(t)| = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}
\]
so that the unit tangent vector is:
\[
\mathbf{T}(t) = \frac{1}{e^t + e^{-t}} (\sqrt{2}, e^t, -e^{-t})
\]
There is a compound fraction there that should probably be simplified:
\[
\frac{1}{e^t + e^{-t}} = \frac{e^t}{e^{2t} + 1}
\]
Multiplying through, we have:
\[
\mathbf{T}(t) = \frac{1}{e^{2t} + 1} \left( \sqrt{2} e^t, e^{2t}, -1 \right)
\]
Recall that the derivative may be computed as:

\[(f(t)u(t))' = f'(t)u(t) + f(t)u'(t)\]

or in this case,

\[T'(t) = -\frac{2e^{2t}}{(e^{2t} + 1)^2} \langle \sqrt{2} e^t, e^{2t}, -1 \rangle + \frac{1}{e^{2t} + 1} \langle \sqrt{2} e^t, 2e^{2t}, 0 \rangle\]

Some algebra later...

\[T'(t) = \frac{1}{(e^{2t} + 1)^2} \langle \sqrt{2} e^t(1 - e^{2t}), 2e^{2t}, 2e^{2t} \rangle\]

And a LOT of algebra later, we can compute \(N(t)\) as:

\[\frac{1}{e^{2t} + 1} \langle 1 - e^{2t}, \sqrt{2} e^t, \sqrt{2} e^t \rangle\]

13.3. 43. There are some shortcuts we can make here. First, some computations and evaluations:

\[r'(t) = \langle 2t, 2t^2, 1 \rangle \Rightarrow |r'(t)| = \sqrt{4t^4 + 4t^2 + 1} = 2t^2 + 1\]

At \(t = 0\), we have the following:

\[r'(1) = \langle 2, 2, 1 \rangle \Rightarrow |r'(1)| = 3\]

Therefore,

\[T(1) = \frac{1}{3} \langle 2, 2, 1 \rangle\]

To compute \(N\), we need \(T\) in terms of \(t\):

\[T(t) = \frac{1}{2t^2 + 1} \langle 2t, 2t^2, 1 \rangle\]

Use the product rule to differentiate:

\[T'(t) = \frac{-4t}{(2t^2 + 1)^2} \langle 2t, 2t^2, 1 \rangle + \frac{1}{2t^2 + 1} \langle 2, 4t, 0 \rangle\]

Now, we only need \(N(1)\), so evaluate \(T'(1)\), then make it a unit vector.

\[T'(1) = \frac{1}{9} \langle -2, 4, -4 \rangle = \frac{2}{9} \langle -1, 2, -2 \rangle\]

To find \(N(1)\) a unit vector, we can really just ignore the \(2/9\) and just make the vector have magnitude 1:

\[N(1) = \frac{1}{3} \langle -1, 2, -2 \rangle\]

Now, the binormal vector is the cross product of \(T\) and \(N\):

\[B(1) = T(1) \times N(1) = \frac{1}{3} \langle -2, 1, 2 \rangle\]
The computations are very similar to the previous problem:
\[ \mathbf{r}'(t) = \langle -\sin(t), \cos(t), \tan(t) \rangle \implies |\mathbf{r}'(t)| = \sqrt{1 + \tan^2(t)} = \sec(t) \]
Note that secant is positive close to \( t = 0 \). Now, compute these at \( t = 0 \) for \( \mathbf{T}(0) \):
\[ \mathbf{T}(0) = \langle 0, 1, 0 \rangle \]
Simplifying before we differentiate \( \mathbf{T} \), we have:
\[ \mathbf{T}(t) = \langle -\sin(t) \cos(t), \cos^2(t), -\sin(t) \rangle \]
so that differentiating,
\[ \mathbf{T}'(t) = \langle \sin^2(t) - \cos^2(t), -2\sin(t) \cos(t), -\cos(t) \rangle \]
Now we can compute \( \mathbf{N}(0) \):
\[ \mathbf{N}(0) = \frac{1}{\sqrt{2}} \langle -1, 0, -1 \rangle \]
Finally, compute the binormal vector using the cross product:
\[ \mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \frac{1}{\sqrt{2}} \langle -1, 0, 1 \rangle \]
13.3, 45. Compute the equations of the normal plane (whose normal vector is \( \mathbf{T} \)) and the osculating plane (whose normal vector is \( \mathbf{B} \)), for the given function at \( t = \pi \). First compute the TNB vectors as before:
\[ \mathbf{r}'(t) = \langle 6 \cos(3t), 1, -6 \sin(3t) \rangle \implies |\mathbf{r}'(t)| = \sqrt{36 \cos^2(3t) + 1 + 36 \sin^2(t)} = \sqrt{37} \]
Therefore,
\[ \mathbf{T}(\pi) = \frac{1}{\sqrt{37}} \langle -6, 1, 0 \rangle \]
And for \( \mathbf{N} \), we need to differentiate \( \mathbf{T} \):
\[ \mathbf{T}'(t) = \frac{1}{\sqrt{37}} \langle -18 \sin(3t), 0, -18 \cos(3t) \rangle \]
and we see that (again, ignore the constant and just make \( \mathbf{N} \) a unit vector):
\[ \mathbf{N}(\pi) = \langle 0, 0, 1 \rangle \]
Finally, the binormal vector is the cross product:
\[ \mathbf{B}(\pi) = \frac{1}{\sqrt{37}} \langle 1, 6, 0 \rangle \]
The planes will intersect the curve at \( t = \pi \), or at the point \( (0, \pi, -2) \), so now we have the normal vectors and the point:
Figure 1: Figure showing the curve, the normal plane (cuts the curve) and the osculating plane (embeds the curve) for Exercise 45, section 13.3.

- The normal plane (we use a nice scalar multiple of $T(\pi)$)
  
  $$-6(x - 0) + 1(y - \pi) + 0(z + 2) = 0 \quad \Rightarrow \quad -6x + y - \pi = 0$$

- The osculating plane uses $B$ as the normal:
  
  $$(x - 0) + 6(y - \pi) + 0(z + 2) = 0 \quad \Rightarrow \quad x + 6y - 6\pi = 0$$

13.3, 46. Same type of computations as 45, but the length is a bit messy:

$$\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle \quad \Rightarrow \quad |\mathbf{r}'(t)| = \sqrt{1 + 4t^2 + 9t^4}$$

Therefore,

$$T(1) = \frac{1}{\sqrt{14}}\langle 1, 2, 3 \rangle$$

Notice that we could stop here to get the equation for the normal plane using the vector $\langle 1, 2, 3 \rangle$:

$$1(x - 1) + 2(y - 1) + 3(z - 1) = 0 \quad \Rightarrow \quad x + 2y + 3z - 6 = 0$$
For $\mathbf{N}$, we need to differentiate $\mathbf{T}$ which is messy. If done correctly, you should get

$$
\mathbf{N}(1) = -\frac{2}{14^{3/2}} (11, 8, -9)
$$

Finally, the binormal vector is the cross product. If you consider the problem, we don’t need to carry along the constants as long as we remember to make the vector a unit vector at the end. In this case,

$$
\langle 1, 2, 3 \rangle \times (11, 8, -9) = (-42, 42, -14)
$$

Factor 14 out and make it a unit vector for $\mathbf{N}$ (or, if we don’t need $\mathbf{N}$, but just the osculating plane, just factor out 14):

$$
\mathbf{N}(1) = \frac{1}{\sqrt{19}} (-3, 3, -1)
$$

and the osculating plane is:

$$
-3(x - 1) + 3(y - 1) - (z - 1) = 0 \implies -3x + 3y + z - 1 = 0
$$

13.3, 49. The normal plane is perpendicular to the tangent vector (either $\mathbf{r}'(t)$ or $\mathbf{T}(t)$):

$$
\mathbf{r}'(t) = (3t^2, 3, 4t^3)
$$

We need to find a $t$ so that $\mathbf{r}'(t)$ is parallel to $(6, 6, -8)$ or factoring 2 out, we need $\mathbf{r}'(t)$ to be parallel to $(3, 3, -4)$.

Now we see that they are parallel by taking $t = -1$, which corresponds to the point $(-1, -3, 1)$.

13.3, 59. A fun problem- First, from the information given, you should get (in angstroms):

$$
\mathbf{r}(t) = (10 \cos(t), 10 \sin(t), 34t/(2\pi))
$$

Then the arc length for one turn:

$$
\int_0^{2\pi} \sqrt{100 + \frac{34^2}{2\pi}} \, dt \approx 71.441177
$$

in angstroms. Multiply that by $2.09 \times 10^8$ to get the full length- about 2.07 meters!

**Section 13.4**

13.4, 15. Antidifferentiate to find the velocity and position vectors, remembering to use the arbitrary constants!

$$
\mathbf{a}(t) = (1, 2, 0) \implies \mathbf{v}(t) = (t + c_1, 2t + c_2, c_3)
$$

With the initial velocity $(0, 0, 1)$, these are the constants:

$$
\mathbf{v}(t) = (t, 2t, 1) \implies \mathbf{r}(t) = \left(\frac{1}{2}t^2 + c_1, t^2 + c_2, t + c_3\right)
$$

With the initial position $(1, 0, 0)$, we get

$$
\mathbf{r}(t) = \left(\frac{1}{2}t^2 + 1, t^2, t\right)
$$
13.4, 19. With $\mathbf{r}(t)$ given, the speed is the magnitude of velocity. We can find the minimum of that by finding the minimum of the square of the magnitude,

$$|\mathbf{r}'(t)|^2 = 4t^2 + 25 + (2t - 16)^2 = 8t^2 - 64t + 281$$

To find the minimum, take the derivative and set to zero:

$$16t - 64 = 0 \implies t = 4$$

Does a minimum occur at $t = 4$ or does a maximum? The second derivative is 16 (positive), so at $t = 4$ we have a minimum- The minimum speed is found by substituting into the formula for the (square of) the speed:

$$\sqrt{153} = 3\sqrt{17}$$

13.4, 21. Information we are given in the problem: Since the force is directed upwards with a magnitude of 20, the force vector is:

$$\mathbf{F}(t) = 20\mathbf{k}$$
Force is also mass times acceleration, so

\[20k = 4a(t) \Rightarrow \vec{a}(t) = 5\vec{k}\]

Now we integrate to find the velocity:

\[v(t) = \int \vec{a}(t) \, dt = \langle c_1, c_2, 5t + c_3 \rangle\]

And \(\vec{v}(0) = \langle 1, -1, 0 \rangle\), so these are our constants, and

\[\vec{v}(t) = \langle 1, -1, 5t \rangle\]

Position is found by integrating velocity,

\[\vec{r}(t) = \left< t + c_1, -t + c_2, \frac{5}{2}t^2 + c_3 \right>\]

The object begins at the origin, so \(\vec{r}(0) = \vec{0}\), therefore the constants are zero and our position function is:

\(< t, t, (5/2)t^2 >\)

13.4, 23. A projectile is fired with an initial speed of 500 meters per second and an angle of elevation of 30 degrees.

Before continuing, use the results of Example 5 (Equation 4), on p. 841 to write:

\[x(t) = 500 \cos(30)t = 250\sqrt{3}t\]

And for the \(y\)-coordinate,

\[y(t) = 500 \sin(30) t - \frac{1}{2}gt^2 = 250t - 9.8 \cdot \frac{1}{2}t^2 = 250t - 4.9t^2\]

Our position vector is therefore \(\vec{r}(t) = \langle 250\sqrt{3}t, 250t - 4.9t^2 \rangle\)

To find the range of the projectile, we determine the times at which the height was zero. These should correspond to the time at launch and the time at landing:

\[250t - 4.9t^2 = 0 \Rightarrow t(250 - 4.9t) = 0 \Rightarrow t = 0, t \approx 51.02\]

At 51.02 seconds, the \(x\)-coordinate was approximately 22,092 meters, or about 22 km.

To find the speed at impact, we substitute \(t = 51.02\) into the expression for \(|\vec{v}(t)||:

\[|\vec{v}(t)| = |\vec{r}'(t)| = \sqrt{250^2 \cdot 3 + (250 - 9.8t)^2}\]

At \(t \approx 51.02\), we should find that the speed was approximately 500 meters per second.

13.4, 25. Similarly (to Exercise 23 and Example 5), if a ball is thrown at an angle of 45 degrees to the ground, we can write:

\[x(t) = \frac{v_0}{\sqrt{2}}t \quad y(t) = \frac{v_0}{\sqrt{2}}\]