Selected Solutions: 3.2

1. Problem 16, 3.2: We are told that \( y = \sin(t^2) \) is a solution- Substitute it into the DE to determine \( p(t) \) and \( q(t) \):

\[
\begin{align*}
y &= \sin(t^2) & y' &= 2t \cos(t^2) & y'' &= 2 \cos(t^2) - 4t^2 \sin(t^2) \\
\end{align*}
\]

Substituting into \( y'' + p(t)y' + q(t)y = 0 \), we get:

\[
\begin{align*}
y'' &= 2 \cos(t^2) - 4t^2 \sin(t^2) \\
p(t)y' &= p(t)2t \cos(t^2) \\
q(t)y &= q(t) \sin(t^2) \\
0 &= (2 + 2tp(t)) \cos(t^2) + (q(t) - 4t^2) \sin(t^2)
\end{align*}
\]

Therefore, if \( \sin(t^2) \) was a solution, \( p(t) = -\frac{1}{t} \) and \( q(t) = 4t^2 \). However, this would make \( p(t) \) not continuous at \( t = 0 \).

2. Problem 18, Sect 3.2:

We’re given the Wronskian and a function \( f(t) = t \). Find the function \( g \):

\[
W(f; g)(t) = \begin{vmatrix} t & g(t) \\ 1 & g'(t) \end{vmatrix} = tg'(t) - g(t) = t^2 e^t
\]

That is a linear differential equation in \( g \):

\[
g'(t) - \frac{1}{t} g(t) = te^t
\]

The integrating factor: \( e^{-\int \frac{1}{t} dt} = \frac{1}{t} \), so:

\[
\left( \frac{g(t)}{t} \right)' = e^t \quad \Rightarrow \quad g(t)/t = e^t + C \quad \Rightarrow \quad g(t) = t(e^t + C)
\]

3. Problem 22, Section 3.2: Find the fundamental set (Theorem 3.2.5) if \( y'' + 4y' + 3y = 0 \).

We know the general solution is \( C_1 e^{-t} + C_2 e^{-3t} \). In Theorem 3.2.5, we construct two solutions that are guaranteed to form a fundamental set:

- \( y_1 \) solves the ODE with initial conditions \( y(1) = 1, y'(1) = 0 \):

\[
\begin{align*}
C_1 e^{-1} + C_2 e^{-3} &= 1 \\
- C_1 e^{-1} - 3C_2 e^{-3} &= 0
\end{align*} \quad \Rightarrow \quad C_1 = \frac{3}{2} e, C_2 = -\frac{1}{2} e^3
\]

Therefore, \( y_1(t) = \frac{3}{2} e^{-t} - \frac{e^3}{2} e^{-3t} \).
• Similarly, $y_2$ solves the ODE with I.C.s: $y(1) = 0, y'(1) = 1$:

$$C_1e^{-t} + C_2e^{-3t} = 0 \quad \Rightarrow \quad C_1 = \frac{e^3}{2}, C_2 = -\frac{e^3}{2}$$

Therefore, $y_2(t) = \frac{e}{2}e^{-t} - \frac{e^3}{2}e^{-3t}$

Even though $y_1$ and $y_2$ look a lot alike, Theorem 3.2.5 guarantees that they are linearly independent, and that they form a fundamental set.

Note that Theorem 3.2.5 is more of a formal result than something we would actually compute with; however, it does give conditions on which we can always guarantee that we can find a fundamental set of solutions.

4. Problem 26, Section 3.2: The verification is straightforward.

Before we consider the question of whether we have a fundamental set, look at where the solutions would be valid.

We will have a discontinuity of $p, q, g$ at the $x-$values where:

$$1 - x \cot(x) = 0 \quad \Rightarrow \quad x \cot(x) = 1 \quad \Rightarrow \quad x \frac{\cos(x)}{\sin(x)} = 1 \quad \Rightarrow \quad x \cos(x) = \sin(x)$$

Therefore, existence and uniqueness is only guaranteed on intervals which avoid these points.

Going to the original question, does $x$ and $\sin(x)$ constitute a fundamental set? Look at the Wronskian:

$$W(x, \sin(x)) = \begin{vmatrix} x & \sin(x) \\ 1 & \cos(x) \end{vmatrix} = x \cos(x) - \sin(x)$$

This looks very familiar! The Wronskian will be non-zero for intervals which also satisfy the existence and uniqueness theorem.

Finally, is the given interval one such example? The expression will not be zero on $0 < x < \pi$ (check this graphically).

5. Problem 27, Section 3.2: Just a couple of notes here. You should find that $y_1, y_3$ do form a fundamental set; $y_2, y_3$ do NOT form a fundamental set.

To show that $y_1, y_4$ do form a fundamental set, notice that, since $y_1, y_2$ do form a fundamental set,

$$y_1y_2' - y_1'y_2 \neq 0 \text{ at } t_0$$

Now form the Wronskian between $y_1$ and $y_4$:

$$W(y_1, y_4) = \begin{vmatrix} y_1 & y_1 + 2y_2 \\ y_1' & y_1' + 2y_2' \end{vmatrix} = y_1y_1' + 2y_1y_2' - y_1'y_2 - 2y_1'y_2 = 2W(y_1, y_2) \neq 0$$

The last set, $y_4, y_5$ does NOT form a fundamental set. You can show that $y_4 = y_1 + 2y_2$, and $y_5 = 2y_4$. 