Selected Problems from Section 3.7

1. Problem 3: \( y'' + 2y' + y = 3e^{-t} \)

Using Variation of Parameters:

\[
y_1 = e^{-t} \quad y_2 = te^{-t} \quad g(t) = 3e^{-t} \quad W = e^{-2t}
\]

so

\[
u_1' = \frac{-3te^{-t}}{e^{-2t}} = -3t \quad \Rightarrow \quad u_1(t) = -\frac{3}{2}t^2
\]

\[
u_2' = \frac{e^{-t}3e^{-t}}{e^{-2t}} = 3 \quad \Rightarrow \quad u_2(t) = 3t
\]

Therefore, the particular solution is \( y_p = u_1 y_1 + u_2 y_2 \):

\[
y_p(t) = -\frac{3}{2}t^2e^{-t} + 3te^{-t} = \frac{3}{2}t^2e^{-t}
\]

Using the method of undetermined coefficients, we would have initially guessed that \( y_p \) was a constant times the exponential function, but we have to multiply by \( t^2 \). Do that, and differentiate to substitute back into the ODE:

\[
y_p = At^2e^{-t} \quad y_p' = A(2t - t^2)e^{-t} \quad y_p'' = A(2 - 4t + t^2)e^{-t}
\]

We end up with \( A = \frac{3}{2} \), which is what we had using Variation of Parameters.

2. Problem 5: \( y'' + y = \tan(t) \)

\[
y_1 = \cos(t) \quad y_2 = \sin(t) \quad g(t) = \tan(t) \quad W = 1
\]

Therefore, we get an integral that can be a bit tricky, but we can simplify the integrand first:

\[
u_1' = -\sin(t) \tan(t) = -\frac{\sin^2(t)}{\cos(t)} = \frac{-1 + \cos^2(t)}{\cos(t)} = -\sec(t) + \cos(t)
\]

so that \( u_1 = -\ln|\sec(t) + \tan(t)| + \sin(t) \).

\[
u_2' = \cos(t) \tan(t) \quad \Rightarrow \quad u_2 = \int \sin(t) \, dt = -\cos(t)
\]

Therefore,

\[
y_p = (-\ln|\sec(t) + \tan(t)| + \sin(t)) \cos(t) - \cos(t) \sin(t) = -\cos(t) \ln|\sec(t) + \tan(t)|
\]
3. Problem 15: We won’t verify that \( y_1, y_2 \) are solutions, but you should. Assuming that they are (remember to put the equation in standard form by dividing by \( t \)):

\[
y_1 = 1 + t \quad y_2 = e^t \quad g(t) = te^{2t} \quad W = te^t
\]

\[
u_1' = \frac{-e^t te^{2t}}{te^t} = -e^{2t} \quad \Rightarrow \quad u_1(t) = -\frac{1}{2} e^{2t}
\]

For \( u_2 \), we’ll integrate by parts:

\[
u_2' = \frac{(1 + t)te^{2t}}{te^t} = (1 + t)e^t \quad \Rightarrow \quad u_2(t) = te^t
\]

The particular solution is \( u_1y_1 + u_2y_2 \) which simplifies to

\[
y_p = \frac{1}{2} e^{2t} (t - 1)
\]

4. Problem 17 comes out nicely as well- Be sure to get standard form before going too far. Use \( u = \ln(x) \) for the integrals.