Summary of Chapter 3

We can think of the chapter as being split into two: General theory, and Computation. First, the general theory.

General Theory, Chapter 3

The goal of the theory was to establish the structure of solutions to the second order DE:

\[ y'' + p(t)y' + q(t)y = g(t) \]

We saw that two functions form a fundamental set of solutions to the homogeneous DE if the Wronskian is not zero (at the initial value of time).


2. Theorems:
   - The Existence and Uniqueness Theorem for \( y'' + p(t)y' + q(t)y = g(t) \): If there is an open interval \( I \) on which \( p, q \) and \( g \) exists, and if \( I \) contains the initial time \( t_0 \), then there exists a unique solution to the IVP, valid on \( I \).
   - Principle of Superposition: If \( L \) is a linear operator, and \( y_1, y_2 \) are two functions so that \( L(y_1) = 0 \) and \( L(y_2) = 0 \), then so does any function of the form \( c_1y_1 + c_2y_2 \).
   - Abel’s Theorem.
     If \( y_1, y_2 \) are solutions to \( y'' + p(t)y' + q(t)y = 0 \), then the Wronskian is either always zero or never zero on the interval for which the solutions are valid.
     That is because the Wronskian may be computed as:
     \[ W(y_1, y_2)(t) = Ce^{-\int p(t) dt} \]
   - The Fundamental Set of Solutions: \( y'' + p(t)y' + q(t)y = 0 \)
     We can guarantee that we can always find a fundamental set of solutions. We did that by appealing to the Existence and Uniqueness Theorem for the following two initial value problems:
     - \( y_1 \) solves \( y'' + p(t)y' + q(t)y = 0 \) with \( y(t_0) = 1, y'(t_0) = 0 \)
     - \( y_2 \) solves \( y'' + p(t)y' + q(t)y = 0 \) with \( y(t_0) = 0, y'(t_0) = 1 \)

3. The Structure of Solutions to \( y'' + p(t)y' + q(t)y = g(t) \), \( y(t_0) = y_0, y'(t_0) = v_0 \)
   Given that \( y_h \) solves the homogeneous equation, and \( y_p \) solves the forced equation, then the general solution to the forced equation is
   \[ y_h + y_p \]
   Or, we can be much more specific:
   Given a fundamental set of solutions to the homogeneous equation, \( y_1, y_2 \), then there is a solution to the initial value problem, written as:
   \[ y(t) = C_1y_1(t) + C_2y_2(t) + y_p(t) \]
   where \( y_p(t) \) solves the non-homogeneous equation.

In fact, if we have:

\[ y'' + p(t)y' + q(t)y = g_1(t) + g_2(t) + \ldots + g_n(t) \]

we can solve by splitting the problem up into smaller problems:
• $y_1, y_2$ form a fundamental set of solutions to the homogeneous equation.
• $y_{p_1}$ solves $y'' + p(t)y' + q(t)y = g_1(t)$
• $y_{p_2}$ solves $y'' + p(t)y' + q(t)y = g_2(t)$
• and so on..
• $y_{p_n}$ solves $y'' + p(t)y' + q(t)y = g_n(t)$

and the full solution is:
$$y(t) = C_1 y_1 + C_2 y_2 + y_{p_1} + y_{p_2} + \ldots + y_{p_n}$$

### Computation of Solutions, Chapter 3

From the theory, we know that every initial value problem:
$$ay'' + by' + cy = g(t) \quad y(t_0) = y_0 \quad y'(t_0) = v_0$$

has a solution that can be expressed as:
$$y(t) = c_1 y_1 + c_2 y_2 + y_p$$

where $y_1, y_2$ form a fundamental set of solutions to the homogeneous equation, and $y_p(t)$ is the (particular) solution to the nonhomogeneous equation.

We first consider the homogeneous ODE:

**Solving $ay'' + by' + cy = 0$**

Form the associated characteristic equation (built by using $y = e^{rt}$ as the ansatz):
$$ar^2 + br + c = 0 \implies r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

so that the solutions depend on the discriminant, $b^2 - 4ac$ in the following way ($y_h$ refers to the solution of the homogeneous equation):

• $b^2 - 4ac > 0 \Rightarrow$ 2 distinct real roots $r_1, r_2$: $y_h(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
• $b^2 - 4ac = 0 \Rightarrow$ one real root $r = -b/2a$: $y_h(t) = e^{-b/2a t} (C_1 + C_2 t)$
• $b^2 - 4ac < 0 \Rightarrow$ 2 complex solutions, $r = \lambda \pm i\mu$: $y_h(t) = e^{\lambda t} (C_1 \cos(\mu t) + C_2 \sin(\mu t))$

**Solving $y'' + p(t)y' + q(t)y = 0$**

Given $y_1(t)$, we can solve for a second linearly independent solution to the homogeneous equation, $y_2$, by one of two methods:

• By use of the Wronskian: There are two ways to compute this,
  - $W(y_1, y_2) = Ce^{-\int p(t) dt}$ (This is from Abel’s Theorem)
  - $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$

  Therefore, these are equal, and $y_2$ is the unknown: $y_1 y_2' - y_2 y_1' = Ce^{-\int p(t) dt}$
• Reduction of order, where $y_2 = v(t)y_1(t)$.  

Finding the particular solution.

Our two methods were: Method of Undetermined Coefficients and Variation of Parameters.

- **Method of Undetermined Coefficients**

  This method is motivated by the observation that, a linear operator of the form
  $$L(y) = ay'' + by' + cy,$$
  acting on certain classes of functions, returns the same class. In summary, the table from the text:

<table>
<thead>
<tr>
<th>if $g_1(t)$ is:</th>
<th>The ansatz $y_p$ is:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_n(t)$</td>
<td>$t^n(a_0 + a_1 t + \ldots + a_n t^n)$</td>
</tr>
<tr>
<td>$P_n(t)e^{at}$</td>
<td>$t^ne^{at}(a_0 + a_1 t + \ldots + a_n t^n)$</td>
</tr>
</tbody>
</table>
  | $P_n(t)e^{at}\sin(\mu t)$ or $\cos(\mu t)$ | $t^ne^{at}((a_0 + a_1 t + \ldots + a_n t^n)\sin(\mu t)$
  |                 | $+(b_0 + b_1 t + \ldots + b_n t^n)\cos(\mu t)$) |

  The $t^n$ term comes from an analysis of the homogeneous part of the solution. That is, multiply by $t$ or $t^2$ so that no term of the ansatz is included as a term of the homogeneous solution.

- **Variation of Parameters:** Given
  $$y'' + p(t)y' + q(t)y = g(t),$$
  with $y_1, y_2$ solutions to the homogeneous equation, we write the ansatz for the particular solution as:
  $$y_p = u_1 y_1 + u_2 y_2$$

  From our analysis, we saw that $u_1, u_2$ were required to solve:

  $$u_1' y_1 + u_2' y_2 = 0$$
  $$u_1' y_1' + u_2' y_2' = g(t)$$

  From which we get the formulas for $u_1'$ and $u_2'$:

  $$u_1' = -\frac{y_2g}{W(y_1, y_2)} \quad u_2' = \frac{y_1 g}{W(y_1, y_2)}$$