Quick Overview: Complex Numbers

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1 Initial Definitions

Definition 1 The complex number \( z \) is defined as:

\[
  z = a + bi \tag{1}
\]

where \( a, b \) are real numbers and \( i = \sqrt{-1} \).

Remarks about the definition:

- Engineers typically use \( j \) instead of \( i \).
- Examples of complex numbers:
  \[ 5 + 2i, \ 3 - \sqrt{2}i, \ 3, \ -5i \]
- Powers of \( i \):
  \[
  \begin{align*}
  i^2 &= -1 & i^3 &= -i \\
  i^4 &= 1 & i^5 &= i \\
  i^6 &= -1 & i^7 &= -i \\
  \vdots
  \end{align*}
  \]
- All real numbers are also complex (by taking \( b = 0 \)).

2 Visualizing Complex Numbers

A complex number is defined by it’s two real numbers. If we have \( z = a + bi \), then:

Definition 2 The real part of \( a + bi \) is \( a \),

\[
  \text{Re}(z) = \text{Re}(a + bi) = a
\]

The imaginary part of \( a + bi \) is \( b \),

\[
  \text{Im}(z) = \text{Im}(a + bi) = b
\]
Figure 1: Visualizing $z = a + bi$ in the complex plane. Shown are the modulus (or length) $r$ and the argument (or angle) $\theta$.

To visualize a complex number, we use the complex plane $\mathbb{C}$, where the horizontal (or $x$-) axis is for the real part, and the vertical axis is for the imaginary part. That is, $a + bi$ is plotted as the point $(a,b)$.

In Figure 1, we can see that it is also possible to represent the point $a + bi$, or $(a,b)$ in polar form, by computing its modulus (or size), and angle (or argument):

$$r = |z| = \sqrt{a^2 + b^2} \quad \theta = \arg(z)$$

We have to be a bit careful defining $\phi$, since there are many ways to write $\phi$ (and we could add multiples of $2\pi$ as well). Typically, the argument of the complex number $z = a + bi$ is defined to be the 4-quadrant “inverse tangent”\(^1\) that returns $-\pi < \theta \leq \pi$.

That is, formally we can define the argument as:

$$\theta = \arg(a + bi) = \begin{cases} 
\tan^{-1}(b/a) & \text{if } a > 0 \\
\tan^{-1}(b/a) + \pi & \text{if } a < 0 \text{ and } b \geq 0 \\
\tan^{-1}(b/a) - \pi & \text{if } a < 0 \text{ and } b < 0 \\
\pi/2 & \text{if } x = 0 \text{ and } y > 0 \\
-\pi/2 & \text{if } x = 0 \text{ and } y < 0 \\
\text{Undefined} & \text{if } x = 0 \text{ and } y = 0
\end{cases}$$

Quad I and IV
Quad II
Quad III
(Upper imag axis)
(Upper imag axis)
The origin

Examples

Find the modulus $r$ and argument $\theta$ for the following numbers (Hint: It is easiest to visualize these in the plane):

- $z = -3$: SOLUTION: $r = 3$ and $\theta = \pi$
- $z = 2i$: SOLUTION: $r = 2$ and $\theta = \pi/2$

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\(^1\)For example, in Maple this special angle is computed as $\text{arctan}(b,a)$, and in Matlab the command is $\text{atan2}(b,a)$.\)
• \( z = -1 + i \): SOLUTION: \( r = \sqrt{2} \) and \( \theta = \tan^{-1}(-1) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4} \)

• \( z = -3 - 2i \) (Numerical approx from Calculator OK):
SOLUTION: \( r = \sqrt{14} \) and \( \theta = \tan^{-1}(2/3) - \pi \approx 0.588 - \pi \approx -2.55 \text{ rad} \)

3 Operations on Complex Numbers

3.1 The Conjugate of a Complex Number
If \( z = a + bi \) is a complex number, then its conjugate, denoted by \( \bar{z} \), is \( a - bi \). For example,
\[
\begin{align*}
  z &= 3 + 5i \Rightarrow \bar{z} = 3 - 5i \\
  z &= i \Rightarrow \bar{z} = -i \\
  z &= 3 \Rightarrow \bar{z} = 3
\end{align*}
\]
Graphically, the conjugate of a complex number is its mirror image across the horizontal axis. If \( z \) has magnitude \( r \) and argument \( \theta \), then \( \bar{z} \) has the same magnitude with a negative argument.

3.2 Addition/Subtraction, Multiplication/Division
To add (or subtract) two complex numbers, add (or subtract) the real parts and the imaginary parts separately:
\[
(a + bi) \pm (c + di) = (a + c) \pm (b + d)i
\]
To multiply, expand it as if you were multiplying polynomials:
\[
(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i
\]
and simplify using \( i^2 = -1 \). Note what happens when you multiply a number by its conjugate:
\[
z\bar{z} = (a + bi)(a - bi) = a^2 - abi + abi - b^2i^2 = a^2 + b^2 = |z|^2
\]
Division by complex numbers \( z, w \): \( \frac{z}{w} \), is defined by translating it to real number division (rationalize the denominator):
\[
\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{z\bar{w}}{|w|^2}
\]
Example:
\[
\frac{1 + 2i}{3 - 5i} = \frac{(1 + 2i)(3 + 5i)}{34} = -\frac{7}{34} + \frac{11}{34}i
\]

4 The Polar Form of Complex Numbers

4.1 Euler’s Formula
Any point on the unit circle can be written as \( (\cos(\theta), \sin(\theta)) \), which corresponds to the complex number \( \cos(\theta) + i\sin(\theta) \). It is possible to show the following directly, but we’ll use it as a definition:

Definition (Euler’s Formula): \( e^{i\theta} = \cos(\theta) + i\sin(\theta) \).
4.2 Polar Form of $a + bi$:

The polar form is defined as:

$$z = re^{i\theta} \quad \text{where} \quad r = |z| = \sqrt{a^2 + b^2} \quad \theta = \arg(z)$$

4.2.1 Examples

Given the complex number in $a + bi$ form, give the polar form, and vice-versa:

1. $z = 2i$ SOLUTION: Since $r = 2$ and $\theta = \pi/2$, $z = 2e^{i\pi/2}$

2. $z = 2e^{-i\pi/3}$

   We recall that $\cos(\pi/3) = 1/2$ and $\sin(\pi/3) = \sqrt{3}/2$, so

   $$z = 2(\cos(-\pi/3) + i\sin(-\pi/3)) = 2(\cos(\pi/3) - i\sin(\pi/3)) = 1 - \sqrt{3}i$$

5 Exponentials and Logs

The logarithm of a complex number is easy to compute if the number is in polar form:

$$\ln(a + bi) = \ln(re^{i\theta}) = \ln(r) + \ln(e^{i\theta}) = \ln(r) + i\theta$$

The logarithm of zero is left undefined (as in the real case). However, we can now compute the log of a negative number:

$$\ln(-1) = \ln(1 \cdot e^{i\pi}) = i\pi \quad \text{or the log of } i : \quad \ln(i) = \ln(1) + \frac{\pi}{2}i = \frac{\pi}{2}i$$

Note that the usual rules of exponentiation and logarithms still apply.

To exponentiate a number, we convert it to multiplication (a trick we used in Calculus when dealing with things like $x^x$):

$$a^b = e^{b\ln(a)}$$

Example, $2^i = e^{i\ln(2)} = \cos(\ln(2)) + i\sin(\ln(2))$

Example: $\sqrt{1 + i} = (1 + i)^{1/2} = (\sqrt{2}e^{i\pi/4})^{1/2} = (2^{1/4})e^{i\pi/8}$

Example: $i^i = e^{i\ln(i)} = e^{i(i\pi/2)} = e^{-\pi/2}$

6 Real Polynomials and Complex Numbers

If $ax^2 + bx + c = 0$, then the solutions come from the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In the past, we only took real roots. Now we can use complex roots. For example, the roots of $x^2 + 1 = 0$ are $x = i$ and $x = -i$.

Check:

$$(x - i)(x + i) = x^2 + xi - xi - i^2 = x^2 + 1$$

Some facts about polynomials when we allow complex roots:
1. An \( n \text{th} \) degree polynomial can always be factored into \( n \) roots. (Unlike if we only have real roots!) This is the **Fundamental Theorem of Algebra**.

2. If \( a + bi \) is a root to a real polynomial, then \( a - bi \) must also be a root. This is sometimes referred to as “roots must come in conjugate pairs”.

### 7 Exercises

1. Suppose the roots to a cubic polynomial are \( a = 3, b = 1 - 2i \) and \( c = 1 + 2i \). Compute \((x - a)(x - b)(x - c)\).

2. Find the roots to \( x^2 - 2x + 10 \). Write them in polar form.

3. Show that:

   \[
   \text{Re}(z) = \frac{z + \bar{z}}{2} \quad \text{Im}(z) = \frac{z - \bar{z}}{2i}
   \]

4. For the following, let \( z_1 = -3 + 2i, z_2 = -4i \)
   
   (a) Compute \( z_1 \bar{z}_2, \frac{z_2}{z_1} \)
   
   (b) Write \( z_1 \) and \( z_2 \) in polar form.

5. In each problem, rewrite each of the following in the form \( a + bi \):

   (a) \( e^{1+2i} \)
   
   (b) \( e^{2-3i} \)
   
   (c) \( e^{i\pi} \)
   
   (d) \( 2^{1-i} \)
   
   (e) \( e^{2-\frac{5}{2}i} \)
   
   (f) \( \pi^i \)

6. For fun, compute the logarithm of each number:

   (a) \( \ln(-3) \)
   
   (b) \( \ln(-1 + i) \)
   
   (c) \( \ln(2e^{3i}) \)