Review Questions: Exam 3

1. What is the ansatz we use for \( y \) in
   - Chapter 6?
   - Section 5.2-5.3?
   - Section 5.4 (for \( x^2y'' + axy' + \beta y = 0 \))?

2. Finish the definitions:
   - The Heaviside function, \( u_c(t) \):
   - The Dirac \( \delta \)-function: \( \delta(t - c) \) (Note: the Dirac function should be defined as a certain limit)
   - Define the convolution: \( (f \ast g)(t) \)
   - A function is of **exponential order** if:

3. Use the definition of the Laplace transform to determine \( \mathcal{L}(f) \):
   (a) \[ f(t) = \begin{cases} 
3, & 0 \leq t < 2 \\
6 - t, & t \geq 2 
\end{cases} \]
   (b) \[ f(t) = \begin{cases} 
e^{-t}, & 0 \leq t < 5 \\
-1, & t \geq 5 
\end{cases} \]

4. Check your answers to Problem 2 by rewriting \( f(t) \) using the step (or Heaviside) function, and use the table to compute the corresponding Laplace transform.

5. Write the following functions in piecewise form (thus removing the Heaviside function):
   (a) \((t + 2)u_2(t) + \sin(t)u_3(t) - (t + 2)u_4(t)\)
   (b) \(\sum_{n=1}^{4} u_{n\pi}(t) \sin(t - n\pi)\)

6. Determine the Laplace transform:
   (a) \(t^2e^{-9t}\)
   (b) \(e^{2t} - t^3 - \sin(5t)\)
   (c) \(t^2y'(t) \) (in terms of \( Y(s) \))
   (d) \(e^{3t} \sin(4t)\)
   (e) \(e^t \delta(t - 3)\)
   (f) \(t^2u_4(t)\)

7. Find the inverse Laplace transform:
   (a) \(\frac{2s - 1}{s^2 - 4s + 6}\)
   (b) \(\frac{7}{(s + 3)^3}\)
   (c) \(\frac{e^{-2s}(4s + 2)}{(s - 1)(s + 2)}\)
   (d) \(\frac{3s - 1}{2s^2 - 8s + 14}\)
   (e) \(\left(e^{-2s} - e^{-3s}\right)\frac{1}{s^2 + s - 6}\)
8. For the following differential equations, solve for $Y(s)$ (the Laplace transform of the solution, $y(t)$). Do not invert the transform.

(a) $y'' + 2y' + 2y = t^2 + 4t, \ y(0) = 0, \ y'(0) = -1$
(b) $y'' + 9y = 10e^{2t}, \ y(0) = -1, \ y'(0) = 5$
(c) $y'' - 4y' + 4y = t^2e^t, \ y(0) = 0, \ y'(0) = 0$

9. Solve the given initial value problems using Laplace transforms:

(a) $2y'' + y' + 2y = \delta(t - 5)$, zero initial conditions.
(b) $y'' + 6y' + 9y = 0, \ y(0) = -3, \ y'(0) = 10$
(c) $y'' - 2y' - 3y = u_1(t), \ y(0) = 0, \ y'(0) = -1$
(d) $y'' + 4y = \delta(t - \pi/2), \ y(0) = 0, \ y'(0) = 1$
(e) $y'' + y = \sum_{k=1}^{\infty} \delta(t - 2k\pi), \ y(0) = y'(0) = 0$. Write your answer in piecewise form.

10. For the following, use Laplace transforms to solve, and leave your answer in the form of a convolution:

(a) $4y'' + 4y' + 17y = g(t) \ \ y(0) = 0, \ y'(0) = 0$
(b) $y'' + y' + \frac{5}{4}y = 1 - u_3(t)$, with $y(0) = 1$ and $y'(0) = -1$.

11. Short Answer:

(a) $\int_0^\infty \sin(3t)\delta(t - \frac{\pi}{2}) \ dt = \text{__________________________}$

(b) Use Laplace transforms to solve the first order DE, thus finding which function has the Dirac function as its derivative:

$$y'(t) = \delta(t - c), \quad y(0) = 0$$

(c) What is the expected radius of convergence for the series expansion of $f(x) = 1/(x^2 + 2x + 5)$ if the series is based at $x_0 = 1$?

(d) Use Laplace transforms to solve for $F(s)$, if

$$f(t) + 2\int_0^t \cos(t - x)f(x) \ dx = e^{-t}$$

(So only solve for the transform of $f(t)$, don’t invert it back).

(e) In order for the Laplace transform of $f$ to exist, $f$ must be ____________________

(f) Can we assume that the solution to: $y'' + p(x)y' + q(x)y = u_3(x)$ is a power series?

(g) Use the table to find the Laplace transform of $e^{-2t}\sinh(\sqrt{3}t)$. (Note: You don’t need the definition the hyperbolic sine to answer this question).

(h) Is $x = 0$ an ordinary point for the differential equation: $xy'' + 3x^2y' + y = 4$?

12. Let $f(t) = t$ and $g(t) = u_2(t)$.
(a) Use the Laplace transform to compute \( f \ast g \).
(b) Verify your answer by computing \( f \ast g \) using the definition.

13. If \( a_0 = 1 \), determine the coefficients \( a_n \) so that
\[
\sum_{n=1}^{\infty} na_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0
\]
Try to identify the series represented by \( \sum_{n=0}^{\infty} a_n x^n \).

14. Write the following as a single sum in the form \( \sum_{k=2}^{\infty} c_k (x-1)^k \) (with a few terms in the front):
\[
\sum_{n=1}^{\infty} n(n-1)a_n (x-1)^{n-2} + x(x-2) \sum_{n=1}^{\infty} na_n (x-1)^{n-1}
\]

15. Characterize ALL (continuous or not) solutions to
\[
y'' + 4y = u_1(t), \quad y(0) = 1, y'(0) = 1
\]

16. Use the table to find an expression for \( \mathcal{L}(ty') \). Use this to convert the following DE into a linear first order DE in \( Y(s) \) (do not solve):
\[
y'' + 3ty' - 6y = 1, y(0) = 0, y'(0) = 0
\]

17. Find the recurrence relation between the coefficients for the power series solutions to the following:
(a) \( 2y'' + xy' + 3y = 0 \), \( x_0 = 0 \).
(b) \( (1-x)y'' + xy' - y = 0 \), \( x_0 = 0 \)
(c) \( y'' - xy' - y = 0 \), \( x_0 = 1 \)

18. Exercises with the table:
(a) Use table entries 5 and 14 to prove the formula for 9.
(b) Show that you can use table entry 19 to find the Laplace transform of \( t^2 \delta(t-3) \) (verify your answer using a property of the \( \delta \) function).
(c) Prove (using the definition of \( \mathcal{L} \)) table entries 12 and 13.
(d) Prove (using the definition of \( \mathcal{L} \)) a formula (similar to 18) for \( \mathcal{L}(y'''(t)) \).

19. Find the first 5 terms of the power series solution to \( e^x y'' + xy = 0 \) if \( y(0) = 1 \) and \( y'(0) = -1 \).

20. Determine a lower bound for the radius of convergence of series solutions about each given point \( x_0 \) for the given differential equation:
\[
(x^2 - 2x + 5)y'' + 4xy' + y = 0 \quad x_0 = 0, \quad x_0 = 3
\]

21. Find the radius of convergence for all of the following, and find the interval of convergence for (b) and (d):
(a) \( \sum_{n=1}^{\infty} \sqrt{n}x^n \)  

(b) \( \sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n+1}}(x+3)^n \)  

(c) \( \sum_{n=1}^{\infty} \frac{n!x^n}{n^n} \) (A little tricky)  

(d) \( \sum_{n=1}^{\infty} \frac{(3x-2)^n}{n5^n} \)

22. Exercises from 5.4, as assigned and if applicable.