Study Guide: Exam 1, Math 244

The exam covers material from Chapters 1 and 2 (up to 2.6), and will be 50 minutes in length. You may not use the text, notes, colleagues or a calculator.

Because a differential equation defines a function (the solution), there are several ways of getting insight into the solution- Graphically, Algebraically, and Numerically. In Chapters 1 and 2, we get a little of the first and third, and a lot of the second.

In summary, the first exam is all about understanding (and solving) first order differential equations: \( y' = f(t, y). \)

**Vocabulary**

- You should know what these terms mean:
  
  - differential equation, ordinary differential equation, partial differential equation, order of a differential equation, linear differential equation, equilibrium solution, isocline, direction field

- Be able to identify the following types of DEs: Linear, separable, homogeneous, autonomous, and Bernoulli.

**The Existence and Uniqueness Theorem**

*Know these!*

1. Linear: \( y' + p(t)y = g(t) \) at \((t_0, y_0)\):

   If \( p, g \) are continuous on an interval \( I \) that contains \( t_0 \), then there exists a unique solution to the initial value problem and that solution is valid for all \( t \) in the interval \( I \).

2. General Case: \( y' = f(t, y) \), \((t_0, y_0)\):

   Let the functions \( f \) and \( f_y \) be continuous in some open rectangle \( R \) containing the point \((t_0, y_0)\). Then there exists an interval about \( t_0 \), \((t_0 - h, t_0 + h)\) contained in \( R \) for which a unique solution to the IVP exists.

   *Side Remark 1:* To determine such a time interval, we must solve the DE.

   *Side Remark 2:* We broke out the theorem in class into two components (existence and uniqueness). You can use either the theorem there or as it stated above.

**Graphical Analysis**

1. Be able to use a direction field to analyze the behavior of solutions to general first order equations. Be able to construct simple direction fields using isoclines.

2. Special Case: **Autonomous DEs:** The main idea here is to be able to graph the phase plot, \( y' = f(y) \) in the \((y, y')\) plane and be able to translate the information from this graph to the direction field, the \((t, y)\) plane.

   Here is a summary of that information:
<table>
<thead>
<tr>
<th>In Phase Diagram:</th>
<th>In Direction Field:</th>
</tr>
</thead>
<tbody>
<tr>
<td>y intercepts</td>
<td>Equilibrium Solutions</td>
</tr>
<tr>
<td>+ to − crossing</td>
<td>Stable Equilibrium</td>
</tr>
<tr>
<td>− to + crossing</td>
<td>Unstable Equilibrium</td>
</tr>
<tr>
<td>y' &gt; 0</td>
<td>y increasing</td>
</tr>
<tr>
<td>y' &lt; 0</td>
<td>y decreasing</td>
</tr>
<tr>
<td>y' and df/dy same sign</td>
<td>y is concave up</td>
</tr>
<tr>
<td>y' and df/dy mixed</td>
<td>y is concave down</td>
</tr>
</tbody>
</table>

Recall that we also looked at a theorem about determining the stability of an equilibrium solution using the sign of $df/dy$, and determining a formula for $y''$ given $y' = f(y)$.

**Analytic Solutions**

- **Linear**: $y' + p(t)y = g(t)$. Use the integrating factor: $e^\int p(t) dt$
- **Separable**: $y' = f(y)g(t)$. Separate variables: $(1/f(y)) dy = g(t) dt$
- **Solve by substitution**:
  - Homogeneous: $\frac{dy}{dx} = F(y/x)$. Substitute $v = y/x$ (and get the expression for $dv/dx$ as well).
  - Bernoulli: $y' + p(t)y = y^n$ Divide by $y^n$, let $w = y^{1-n}$ and it becomes linear.

  **NOTE**: I’ll give a hint for these if I want you to solve one (versus identity one).

- **Exact**: $M(x, y) + N(x, y) \frac{dy}{dx}$, where $N_x = M_y$.  
  Solution: Set $f_x(x, y) = M(x, y)$. Integrate w/r to $x$. Check that $f_y = N(x, y)$, and add a function of $y$ if necessary.

  **NOTE**: I’ll give an integrating factor, if necessary. You should be able to derive equations that define the integrating factor, as done in class and on pages 98-99. That is, if you look in the book, see if you can figure out how Equation 27 on pg. 99 was derived.

**Models**

Be familiar with (be able to construct) the following models:

- For any physics problems, values of constants (like $g$) would be given to you.

**Euler’s Method**

This is the underlying method to many numerical techniques for solving a differential equation. Be able to derive the formula (as done on p. 103), and be able to compute 1-2 iterations by hand (for the exam). For the real world, it is also beneficial to see if you can program the method on a computer, but we’ll wait to do that.