Solutions: Section 2.1

1. Problem 1: See the Maple worksheet to get the direction field. You should see that it looks like all solutions are approaching some curve (maybe a line?) as $t \to \infty$. To be more analytic, let us solve the DE using the Method of Integrating Factors.

\[ y' + 3y = t + e^{-2t} \Rightarrow e^{3t}(y' + 3y) = e^{3t}(t + e^{-2t}) \Rightarrow (e^{3t}y(t))' = te^{3t} + e^t \]

Integrate both sides Hint: We need to use “integration by parts” to integrate $te^{3t}$.

Using a table as in class:

<table>
<thead>
<tr>
<th>$+ t$</th>
<th>$e^{3t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$- 1$</td>
<td>$(1/3)e^{3t}$</td>
</tr>
<tr>
<td>$+ 0$</td>
<td>$(1/9)e^{3t}$</td>
</tr>
</tbody>
</table>

\[ \Rightarrow \int te^{3t} \, dt = \frac{1}{3}e^{3t} - \frac{1}{9}e^{3t} \]

Putting it all together,

\[ e^{3t}y(t) = \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} + e^t + C \]

so that

\[ y(t) = \frac{1}{3}t - \frac{1}{9} + \frac{1}{e^{-2t}} + C e^{3t} \]

Notice that the last two terms go to zero as $t \to \infty$, so we see that $y(t)$ does approach a line:

\[ \frac{1}{3}t - \frac{1}{9} \]

as $t \to \infty$.

2. Problem 3: See Maple for the direction field. Very similar situation to Problem 1. Let’s go ahead and solve:

\[ y' + y = te^{-t} + 1 \]

Multiply both sides by $e^{\int p(t) \, dt} = e^t$:

\[ e^t(y' + y) = t + e^t \Rightarrow (e^ty(t))' = t + e^t \]

Integrate both sides:

\[ e^ty(t) = \frac{1}{2}t^2 + e^t + C \Rightarrow y(t) = \frac{1}{2}t^2e^{-t} + 1 + Ce^{-t} \]

This could be written as:

\[ y(t) = 1 + \frac{t^2}{2e^t} + \frac{C}{e^t} \]

so that it is clear that, as $t \to \infty$, $y(t) \to 1$, which we also see in the direction field.
3. (Extra Practice, not assigned) Problem 11: See Maple for the direction field, where it looks like all solutions are approaching some periodic function as \( t \to \infty \). Let’s solve it:

\[ y' + y = 5 \sin(2t) \]

As in the last exercise, multiply both sides by \( e^t \):

\[ e^t(y' + y) = 5e^t \sin(2t) \quad \Rightarrow \quad (e^t y(t))' = 5e^t \sin(2t) \]

To integrate the right-hand-side of this equation, we will need to use integration by parts twice. In tabular form:

| + | \( e^t \) | \( \sin(2t) \) |
| - | \( e^t \) | \(-\frac{1}{2} \cos(2t) \) |
| + | \( e^t \) | \(-\frac{1}{4} \sin(2t) \) |

\[ \int e^t \sin(2t) \, dt = -\frac{1}{2} e^t \cos(2t) + \frac{1}{4} e^t \sin(2t) - \frac{1}{4} \int e^t \sin(2t) \, dt \]

Add the last integral to the left:

\[ \frac{5}{4} \int e^t \sin(2t) \, dt = -\frac{1}{2} e^t \cos(2t) + \frac{1}{4} e^t \sin(2t) \]

so that:

\[ \int e^t \sin(2t) \, dt = -\frac{2}{5} e^t \cos(2t) + \frac{1}{5} e^t \sin(2t) + C_1 \]

Going back to the differential equation,

\[ e^t y(t) = -2e^t \cos(2t) + e^t \sin(2t) + C_2 \]

so that the general solution is:

\[ y(t) = -2 \cos(2t) + \sin(2t) + C_2 e^{-t} \]

We see that, as \( t \to \infty \), \( y(t) \) does indeed go to a periodic function.

**In problems 13, 15, 16, solve the IVP.** For these problems, I will leave the details out, but I will give the integrating factor. Be sure to ask in class if you’re not sure how to solve them!

4. Problem 13: (You’ll need to integrate by parts!)

\[ y' - y = 2te^{2t} \quad e^{\int p(t) \, dt} = e^{-t} \]

\[ y(t) = e^{2t}(2t - 2) + 3e^t \]

5. Problem 15:

\[ ty' + 2y = t^2 - t + 1 \]
Be sure to put in standard form before solving:

\[ y' + \frac{2}{t} y = t - 1 + \frac{1}{t} \quad e^{\int p(t) \, dt} = t^2 \]

and

\[ y(t) = \frac{1}{4} t^2 - \frac{1}{3} t + \frac{1}{2} + \frac{1}{12t^2} \]

6. Problem 16: In this problem, the integrating factor is again \( t^2 \):

\[ y' + \frac{2}{t} \cdot y = \frac{\cos(t)}{t^2} \quad \Rightarrow \quad y(t) = \frac{\sin(t)}{t^2} \]

7. Problem 30: Solve the IVP:

\[ y' - y = 1 + 3 \sin(t) \quad y(0) = y_0 \]

The integrating factor is \( e^{-t} \), so that we get:

\[ (ye^{-t})' = e^{-t} + 3e^{-t} \sin(t) \]

Integrate both sides. The term on the far right side of the equation requires integration by parts using a table (differentiate the middle column, antidifferentiate the last column):

\[
\begin{array}{|c|c|c|}
\hline
\text{Function} & \text{Derivative} & \text{Integral} \\
\hline
\sin(t) & \cos(t) & e^{-t} \\
\cos(t) & -\sin(t) & -e^{-t} \\
\hline
\end{array}
\]

so that

\[
\int e^{-t} \sin(t) \, dt = -e^{-t} \cos(t) - e^{-t} \sin(t) - \int e^{-t} \sin(t) \, dt
\]

Add the last integral to both sides and divide:

\[ 2 \int e^{-t} \sin(t) \, dt = -e^{-t} \cos(t) - e^{-t} \sin(t) \]

Therefore,

\[ \int e^{-t} \sin(t) \, dt = -\frac{1}{2} e^{-t} (\cos(t) + \sin(t)) \]

Going back to the DE, we have (remember to multiply by 3):

\[ ye^{-t} = -e^{-t} - \frac{3}{2} e^{-t} (\cos(t) + \sin(t)) + C \]

so that

\[ y = -1 - \frac{3}{2} (\cos(t) + \sin(t)) + Ce^t \]
Therefore, the general solution is (using the initial condition):

\[ y(t) = -1 - \frac{3}{2} (\cos(t) + \sin(t)) + \left(\frac{5}{2} + y_0\right) e^t \]

To keep the solution finite (or bounded) as \( t \to \infty \), we must find \( y_0 \) so that the exponential term drops out. This means that \( y_0 = -5/2 \).

**Problems 34-36:** We’re going backwards from a desired solution to the differential equation. We notice a couple of things about our previous work. If we want our solution to approach a given function \( g(t) \), then we might write the full solution as

\[ y(t) = g(t) + Ce^{-t} \]

Then, \( y' = g'(t) - Ce^{-t} \), and \( y' + y = g(t) - g'(t) \) is the differential equation.

8. For Problem 34, we want \( y(t) \to 3 \) as \( t \to \infty \). Here are two possible ways of proceeding:

- Suppose \( y(t) = 3 + Ce^{-t} \) (so it looks a lot like the solutions we got for the previous HW problems). Then \( y' = -Ce^{-t} \), and we see that:

  \[ y' + y = 3 \]

  (I’ll leave the verification to you).

- As another possible approach, we could take:

  \[ y(t) = 3 + \frac{C}{t^2} \]

  Now, \( y' = -2C/t^3 \). We see that if we take \( ty' \) and add it to \( 2y \), the terms with \( C \) cancel and we’re left with 6. Therefore, the ODE is:

  \[ ty' + 2y = 6 \]

  (I’ll leave the verification to you).

9. Problem 35 is similar. There are many ways of constructing such a differential equation. It’s easiest to start with a desired solution.

If we would like \( y(t) = 3 - t + Ce^{-3t} \), then \( y' = -1 - 3Ce^{-3t} \), and:

\[ y' + 3y = 8 - 3t \]