Solutions: Section 2.2

1. Problem 1: Give the general solution: \( y' = \frac{x^2}{y} \)

\[ y \, dy = \frac{x^2}{y} \, dx \quad \Rightarrow \quad \frac{1}{2} y^2 = \frac{1}{3} x^3 + C \]

2. Problem 3: Give the general solution to \( y' + y^2 \sin(x) = 0 \).

First write in standard form:

\[ \frac{dy}{dx} = -y^2 \sin(x) \quad \Rightarrow \quad -\frac{1}{y^2} \, dy = \sin(x) \, dx \]

Before going any further, notice that we have divided by \( y \), so we need to say that this is value as long as \( y(x) \neq 0 \). In fact, we see that the function \( y(x) = 0 \) IS a possible solution.

With that restriction in mind, we proceed by integrating both sides to get:

\[ \frac{1}{y} = -\cos(x) + C \quad \Rightarrow \quad y = \frac{1}{C - \cos(x)} \]

3. Problem 5 Hint: To integrate \( \cos^2(x) \), use the identity

\[ \cos^2(x) = \frac{1 + \cos(2x)}{2} \]

4. Problem 7: Give the general solution:

\[ \frac{dy}{dx} = \frac{x - e^{-x}}{y + e^y} \]

First, note that \( dy/dx \) exists as long as \( y \neq -e^y \). With that requirement, we can proceed:

\[ (y + e^y) \, dy = (x + e^{-x}) \, dx \]

Integrating, we get:

\[ \frac{1}{2} y^2 + e^y = \frac{1}{2} x^2 - e^{-x} + C \]

In this case, we cannot algebraically isolate \( y \), so we’ll leave our answer in this form (we could multiply by two).

5. Problem 9: Let \( y' = (1 - 2x)y^2 \), \( y(0) = -1/6 \).

First, we find the solution. Before we divide by \( y \), we should make the note that \( y \neq 0 \). We also see that \( y(x) = 0 \) is a possible solution (although NOT a solution that satisfies the initial condition).
Now solve:
\[ \int y^{-2} \, dy = \int (1 - 2x) \, dx \quad \Rightarrow \quad -y^{-1} = x - x^2 + C \]
Solve for the initial value:
\[ 6 = 0 + C \Rightarrow C = 6 \]
The solution is (solve for \( y \)):
\[ y(x) = \frac{1}{x^2 - x - 6} = \frac{1}{(x - 3)(x + 2)} \]
The solution is valid only on \(-2 < x < 3\), and we could plot this by hand (also see the Maple worksheet).

6. Problem 11: \( x \, dx + ye^{-x} \, dy = 0 \), \( y(0) = 1 \)
To solve, first get into a standard form, multiplying by \( e^x \), and integrate (integration by parts for the right hand side):
\[ \int y \, dy = - \int xe^x \, dx \quad \Rightarrow \quad \frac{1}{2}y^2 = -xe^x + e^x + C \]
We could solve for the constant before isolating \( y \):
\[ \frac{1}{2} = 0 + 1 + C \quad C = -\frac{1}{2} \]
Now solve for \( y \):
\[ y^2 = 2e^x(x - 1) - \frac{1}{2} \]
and take the positive root, since \( y(0) = +1 \).
\[ y = \sqrt{2e^x(1 - x) - 1} \]
The solution exists as long as:
\[ 2e^x(1 - x) - 1 \geq 0 \]
We use Maple to solve where this is equal to zero (see the Worksheet online). From that, we see that \(-1.678 \leq x \leq 0.768\)

7. Problem 14:
\[ \frac{dy}{dx} = xy^3(1 + x^2)^{-1/2} \quad y(0) = 1 \]
Since we’ll divide by \( y \), we look at the case where \( y = 0 \). We see that it is a possible solution, but not for this initial value, therefore, \( y \neq 0 \):
\[ \int y^{-3} \, dy = \int \frac{x}{\sqrt{x^2 + 1}} \, dx \]
To integrate the right side of the equation, let $u = x^2 + 1$ and integrate using a $u, du$ substitution. Doing that, we get:

$$-rac{1}{2}y^{-2} = \sqrt{x^2 + 1} + C \quad \Rightarrow \quad \frac{1}{y^2} = C_2 - 2\sqrt{x^2 + 1}$$

We could solve for the constant now: $1 = C_2 - 2$, so $C = 3$. Solve for $y$:

$$y(x) = \frac{1}{\sqrt{3 - 2\sqrt{x^2 + 1}}}$$

where we take the positive root since the initial condition was positive.

The solution will exist as long as the denominator is not zero. Solving,

$$3 - 2\sqrt{x^2 + 1} = 0 \quad \sqrt{x^2 + 1} = 3/2 \quad x = \pm \frac{\sqrt{5}}{2}$$

The solution is valid for $-\frac{\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$. See Maple for the plot.

8. Problem 16:

$$\frac{dy}{dx} = \frac{x(x^2 + 1)}{4y^3} \quad y(0) = -\frac{1}{\sqrt{2}}$$

First, we notice that $y \neq 0$. Now separate the variables and integrate:

$$y^4 = \frac{1}{4}x^4 + \frac{1}{2}x^2 + C$$

This might be a good time to solve for $C$: $C = 1/4$, so:

$$y^4 = \frac{1}{4}x^4 + \frac{1}{2}x^2 + \frac{1}{4}$$

The right side of the equation seems to be a nice form. Try some algebra to simplify it:

$$\frac{1}{4}(x^4 + 2x^2 + 1) = \frac{1}{4}(x^2 + 1)^2$$

Now we can write the solution:

$$y^4 = \frac{1}{4}(x^2 + 1)^2 \quad \Rightarrow \quad y = -\frac{1}{\sqrt{2}}\sqrt{x^2 + 1}$$

This solution exists for all $x$ (it is the bottom half of a hyperbola- see the Maple plot).

9. Problem 20: $y^2\sqrt{1 - x^2} dy = \sin^1(x) dx$ with $y(0) = 1$.

To put into standard form, we’ll be dividing so that $x \neq \pm 1$. In that case,

$$\int y^2 dy = \int \frac{\sin^{-1}(x)}{\sqrt{1 - x^2}} dx$$
The right side of the equation is all set up for a \( u, du \) substitution, with \( u = \sin^{-1}(x), du = \frac{1}{\sqrt{x^2 - 1}} \) \( dx \):

\[
\frac{1}{3}y^3 = \frac{1}{2}(\arcsin(x))^2 + C
\]

Solve for \( C \), \( \frac{1}{3} = 0 + C \) so that:

\[
\frac{1}{3}y^3 = \frac{1}{2} \arcsin^2(x) + \frac{1}{3}
\]

Now,

\[
y(x) = \sqrt[3]{\frac{3}{2} \arcsin^2(x) + 1}
\]

The domain of the inverse sine is: \(-1 \leq x \leq 1\). However, we needed to exclude the endpoints. Therefore, the domain is:

\(-1 < x < 1\)

**For Problems 31 and 35:** We have a new class of differential equation called **homogeneous**. The idea is that the first order DE:

\[
y' = f(x, y) = F\left(\frac{y}{x}\right) \equiv F(v)
\]

Here, we substitute \( v = y/x \) and see what we get- The hard part is to make the substitution for \( y' \)- Notice that \( vx = y \), so \( y' = v'x + v \). Substituting, we have:

\[
y' = F(y/x) \quad \Rightarrow \quad v'x + v = F(v)
\]

which is always a separable equation.

10. Problem 31:

\[
\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2
\]

Make the substitutions: \( v = y/x \) and \( y' = v'x + v \):

\[
v'x + v = 1 + v + v^2 \quad \Rightarrow \quad x \frac{dv}{dx} = 1 + v^2 \quad \Rightarrow \quad \frac{dv}{1 + v^2} = \frac{dx}{x}
\]

Integrate both sides to get \( \tan^{-1}(v) = \ln |x| + C \), and now we’ll see if we can solve for \( y \):

\[
\tan^{-1}\left(\frac{y}{x}\right) = \ln |x| + C \quad \Rightarrow \quad y = x \tan(\ln |x| + C)
\]

We have to be a bit careful about the domain for this function- Recall that \( y = \tan(x) \) is invertible only if we restrict \(-\pi/2 < x < \pi/2\) (and \( y \in \mathbb{R} \)). In this case, that means

\[
-\frac{\pi}{2} < \ln |x| + C < \frac{\pi}{2} \quad \Rightarrow \quad -C - \frac{\pi}{2} < \ln |x| < -C + \frac{\pi}{2}
\]

Exponentiating,

\[
e^{-C}e^{-\pi/2} < |x| < e^{-C}e^{\pi/2}
\]

We might go ahead and drop the absolute value at this point.
11. Problem 35: Similar to 31,
\[
\frac{dy}{dx} = \frac{x+3y}{x-y} = \frac{1+3(y/x)}{1-(y/x)}
\]
Substitue again, \(v = y/x\), or \(y = xv\), so \(y' = v'x + v\):
\[
v'x + v = \frac{1+3v}{1-v} \quad \Rightarrow \quad v'x = \frac{-v(1-v)}{1-v} + \frac{1+3v}{1-v} = \frac{v^2 + 2v + 1}{1-v} = \frac{(1+v)^2}{1-v}
\]
Now, let \(u = 1 + v\) (so \(v = u - 1\)), and \(du = dv\):
\[
\int \frac{1-v}{(1+v)^2} dv = \int \frac{dx}{x} \quad \Rightarrow \quad \int \frac{2-u}{u^2} = \ln |x|+C \quad \Rightarrow \quad -2(1+v)^{-1} - \ln |1+v| = \ln |x|+C
\]
Backsubstitute for \(v\) (and simplify):
\[
-\frac{2x}{x+y} - (\ln |x+y| - \ln |x|) = \ln |x| + C \quad \Rightarrow \quad \frac{2x}{x+y} + \ln |x+y| = C_2
\]
This solution is valid as long as \(y \neq -x\). Is the function \(y = -x\) a solution as well? Substitute into the DE, with \(y' = -1\), we see that:
\[
\frac{x+3y}{x-y} = \frac{-2x}{2x} = -1
\]
so indeed it is.

12. Problem 36: Similar- Check your answer with Wolfram Alpha!