Solutions: Section 2.5

1. Problem 1: Given \( \frac{dy}{dt} = ay + by^2 = y(a + by) \) with \( a, b > 0 \). For the more general case, we will let \( y_0 \) be any real number.

Always look for the equilibria first! In this case,

\[
y(a + by) = 0 \quad \Rightarrow \quad y = 0 \text{ or } y = -b/a
\]

To make the phase plot (graph of \( y' \) versus \( y \)), we note that \( ay + by^2 \) is a parabola opening upwards, and it intersects the \( y \)-axis at the equilibria, \( y = 0 \) and \( y = -b/a \). From this graph, we see that \( y = 0 \) is an unstable equilibrium, and \( y = -b/a \) is stable.

2. Problem 3: Given \( \frac{dy}{dt} = y(y-1)(y-2) \), and let \( y_0 \) be any real number (the more general case).

Then the phase plot is a cubic function going through the equilibria at \( y = 0, y = 1, y = 2 \).

3. Problem 7: With the DE,

\[
\frac{dy}{dt} = k(1 - y)^2
\]

the only equilibrium solution is: \( k(1 - y)^2 = 0 \Rightarrow y = 1 \). Graphing this as \( y' \) versus \( y \), we get an upward parabola whose vertex is lying on the \( y \)-axis at \( y = 1 \).

For part (b), see the graph.
For part (c), the DE is separable:

\[
\int \frac{1}{(1-y)^2} \, dy = \int k \, dt \quad \Rightarrow \quad \frac{1}{1-y} = kt + C
\]

(Use \(u, du\) substitution for the integral on the left side of the equation). At this stage, we might as well solve for the arbitrary constant:

\[
\frac{1}{1-y_0} = 0 + C
\]

This is valid as long as \(y_0 \neq 1\). In the case that \(y_0 = 1\), the solution is \(y(t) = 1\) (the equilibrium solution).

Solving for \(y\),

\[
1 - y = \frac{1}{kt + C} \quad \Rightarrow \quad y = 1 - \frac{1}{kt + \frac{1}{1-y_0}}
\]

Let us analyze this last equation: If \(\frac{1}{1-y_0} > 0\), then as \(t \to \infty\), \(kt + \frac{1}{1-y_0} \to \infty\), so \(y(t) \to 1\). Therefore, if \(y_0 < 1\), \(y(t) \to 1\) as \(t \to \infty\) (as expected from the phase plot and direction field).

On the other hand, consider the case when \(y_0 > 1\) (the case when \(y_0 = 1\) gave an equilibrium solution). In this case, \(\frac{1}{1-y_0}\) is negative, which means that there will be a vertical asymptote in positive time (also see figure below)

\[
t = -\frac{1}{k(1-y_0)}
\]

From our phase plot, we expect solutions with \(y_0 > 1\) to go to \(+\infty\)- Does that occur algebraically?

\[
y(t) = 1 - \frac{1}{kt + \frac{1}{1-y_0}} = 1 - \frac{1}{t + \frac{1}{k(1-y_0)}}
\]

so we see that the denominator is approaching zero from the left, so that \(y(t) \to +\infty\) as \(t \to -1/(k(1-y_0))\) from the left.

4. Exercises 8, 10, 11 are in the Figure below.
5. Exercise 14: It is OK to argue this graphically, as we did in class. In particular, you
should be able to draw a function so that \( f(y_0) = 0 \) and \( f'(y_0) > 0 \) (or \( f'(y_0) < 0 \)).

6. Problem 22: Please be sure to read the description carefully- Nice intro to epidemiology.

(a) The equilibria are at \( y = 0 \) and \( y = 1 \). The phase plot of \( y' = \alpha y(1 - y) \) is a parabola opening downward. A sketch of the phase plot shows that \( y = 0 \) is unstable and \( y = 1 \) is stable.

(b) To solve this, we’ll need to use partial fraction decomposition:

\[
\frac{1}{y(1-y)} \, dy = \alpha \, dt \Rightarrow \int \frac{1}{y} + \frac{1}{1-y} \, dy = \alpha t + C \Rightarrow \ln |y| - \ln |1-y| = \alpha t + C
\]

so that

\[
\ln \left| \frac{y}{1-y} \right| = \alpha t + C \Rightarrow \frac{y}{1-y} = Ae^{\alpha t}
\]

Solving for \( A \), \( y_0/(1 - y_0) = A \). Keep this in mind, and let’s solve for \( y \) first:

\[
y(t) = \frac{A e^{\alpha t}}{1 + A e^{\alpha t}}
\]

We will want to analyze what happens as \( t \to \infty \), so it will be more convenient to divide numerator and denominator by \( A e^{\alpha t} \):

\[
y(t) = \frac{1}{A e^{-\alpha t} + 1} = \frac{1}{\frac{1-y_0}{y_0} e^{-\alpha t} + 1}
\]

This solution is valid as long as \( y_0 \neq 0 \) and \( y_0 \neq 1 \). In those cases, our solutions are the equilibrium solutions, \( y(t) = 0 \) and \( y(t) = 1 \). Now let us analyze the behavior of \( y(t) \).

We see that, as \( t \to \infty \), \( y(t) \to 1 \). But this is not the end of the story: If a solution begins with \( y_0 < 0 \), for example, we know that the solution CANNOT approach 1 as \( t \to \infty \), because that would mean it would have to cross \( y(t) = 0 \) (and solutions cannot intersect by the E& U Theorem).

*The following is a much more detailed analysis than what was expected in the homework problem- However, read through it to see exactly what the behavior of all solutions looks like.*

The only point that makes us pause is the denominator. Set it to zero and solve:

\[
1 - \frac{y_0}{y_0} e^{-\alpha t} = -1 \Rightarrow e^{-\alpha t} = \frac{y_0}{y_0 - 1} \Rightarrow t = -\frac{1}{\alpha} \cdot \ln \left( \frac{y_0}{y_0 - 1} \right)
\]

Alternatively,

\[
t = \frac{1}{\alpha} \cdot \ln \left( \frac{y_0 - 1}{y_0} \right) = \frac{1}{\alpha} \cdot \ln \left( 1 - \frac{1}{y_0} \right)
\]

The reason this is a nice way of analyzing \( t \):
• If $y_0 > 1$, then we will be taking the log of a number less than 1 (which gives a negative value). In this case, $t$ is negative and our solution $y(t)$ is valid for all $t > (1/\alpha) \ln(1 - (1/y_0))$, and $y(t) \to 1$ as $t \to \infty$.

• If $0 < y_0 < 1$, this denominator is never zero (no solution for $t$ in the real numbers). In this case, $y(t)$ is valid for ALL $t$ (not just positive), and again the limit as $t \to \infty$ is 1.

• If $y_0 < 0$, then the solution is valid for:

$$-\infty < t < \frac{1}{\alpha} \ln \left(1 - \frac{1}{y_0}\right)$$

so that $y(t)$ has a vertical asymptote on the positive $t$ axis. In this case, it is not appropriate to take the limit as $t \to \infty$.

7. Problem 23:
First solve $y' = -\beta y$, which is $y(t) = y_0 e^{-\beta t}$.

**NOTE:** There is a misprint in Problem 23, in defining $dx/dt$. The disease spreads (or INCREASES) at a rate proportional to the number of carrier-susceptible interactions ($x$- and $y$- interactions), which means that the constant in front should be POSITIVE.

We are told to substitute this into the DE:

$$\frac{dx}{dt} = +\alpha xy = \alpha x \left(y_0 e^{-\beta t}\right)$$

Solve this separable equation for $x(t)$:

$$\int \frac{1}{x} \, dx = \alpha y_0 \int e^{-\beta t} \, dt \quad \Rightarrow \quad \ln |x| = -\frac{\alpha \cdot y_0}{\beta} e^{-\beta t} + C$$

Solving for the initial value,

$$C = \ln |x_0| + \frac{\alpha \cdot y_0}{\beta}$$

so that:

$$\ln |x| = \frac{\alpha \cdot y_0}{\beta} \left(1 - e^{-\beta t}\right) + \ln |x_0|$$

Finally, exponentiating both sides:

$$x(t) = x_0 e^{\frac{\alpha \cdot y_0}{\beta} (1 - e^{-\beta t})}$$

And the limit as $t \to \infty$ of $x(t)$ is $x_0 e^{\frac{\alpha \cdot y_0}{\beta}}$
8. We can plod through the questions in 24, but I hope you’re asking yourself what it is we’re doing: The text is getting to a “normalized” model of the disease, where at time 0 none of the population has the disease \( z(0) = 1.00 \) or \( z(0) = 100\% \), then as time goes on, we’re modeling the percentage of the population that has not yet been exposed to smallpox- That is,

\[
z(t) = \frac{\text{Number of people who have not been exposed to smallpox at time } t}{\text{Number of people who are (still) alive at time } t} = \frac{x(t)}{n(t)}
\]

This is an interesting way of doing the modeling, since we are focused on a single “cohort”.

For our model, the susceptible population will only decline either due to exposure to smallpox (\( \beta \) is the exposure rate, \( \nu \) is the death rate. \textit{Side Remark: The greek symbol } \( \nu \text{ is read as “nu”, or “noo”} \) or death from something else:

\[
dx dt = -(\text{Exposure rate-Smallpox}) - (\text{Death rate from other})
\]

The constants are typically given as proportions- That is, the overall exposure rate to smallpox would be \( \beta x(t) \), and we’re told that the death rate will be \( \mu(t)x(t) \). Putting these together gives us the text’s equation:

\[
dx dt = -(\beta + \mu)x
\]

Now, we might notice that since \( \nu \) is the death rate (as a proportion) from smallpox, and \( \beta \) is the exposure rate, then the overall death rate in the population due to smallpox will be \( \nu \beta x(t) \). Similarly, we need to take away the population that has died from other causes, \( \mu(t)n(t) \) (recall that \( n(t) \) is the number of people alive at time \( t \)). Now we have the DE for \( n(t) \):

\[
dn dt = -\nu \beta x(t) - \mu(t)n(t) = -\nu \beta x - \mu n
\]

Now we get to the questions:

(a) Let \( z = x/n \). Then

\[
\frac{dz}{dt} = \frac{x' n - x n'}{n^2} = -\frac{\beta xn - \mu xn - x(-\nu \beta x - \mu n)}{n^2} = -\beta z(1 - \nu z)
\]

And, since \( x(0) = n(0) \) (everyone is alive and susceptible at time 0), then \( z(0) = 1 \) (or 100%)

(b) To solve the DE, we see that

\[
\int \frac{1}{z(1 - \nu z)} dz = \int -\beta dt
\]
So that, using partial fractions on the left, we get

\[ z(t) = \frac{1}{(1 - \nu)e^{\beta t} + \nu} \]

Using the suggested values of \( \nu = \beta = 1/8 \) and \( t = 20 \), we get \( z \approx 0.093 \), so after 20 years, only about 9.2% of the population remain unexposed to smallpox.

9. Problem 25: The basic idea behind problems 25 and 26 is that there is a new parameter, \( a \). By changing this parameter, we can change the number and type of the equilibrium solutions.

In Problem 25, the equilibrium solutions are given by:

\[
\frac{dy}{dt} = 0 \Rightarrow a - y^2 = 0 \quad \Rightarrow \quad y = \pm \sqrt{a}
\]

Graphically in the phase plot, \( y' = -y^2 \) is an upside down parabola, and \( -y^2 + a \) simply translates the parabola up and down.

Therefore, in words:
- If \( a < 0 \), we have no equilibrium solutions.
- If \( a = 0 \), we have a single equilibrium solution at \( a = 0 \), and it is semistable. Since \( y' \) is always negative (and zero at \( y = 0 \)), in the direction field, solutions that begin above \( y_0 = 0 \) decrease to zero, and solutions that begin below \( y_0 = 0 \) decrease to negative infinity.
- If \( a > 0 \), we have two equilibrium solutions (at \( \sqrt{a} \) and \( -\sqrt{a} \)). The positive root is a stable equilibrium, and the negative root is an unstable equilibrium.

We can summarize this graphically in Figure 2.5.10 on page 93.

10. Problem 26: Finding the equilibrium:

\[ y(a - y^2) = 0 \]

We see that \( y(t) = 0 \) is ALWAYS an equilibrium solution for any value of \( a \). The other solutions will be the same as before (we’ll have to re-do our stability analysis):
- If \( a < 0 \), the only equilibrium is \( y(t) = 0 \), and this is stable.
- If \( a = 0 \), same situation.
- If \( a > 0 \), two new equilibria appear, \( y(t) = \pm \sqrt{a} \). Now, \( y(t) = 0 \) switches stability (it is now unstable), and the two new equilibria, \( y(t) = \pm \sqrt{a} \) are both stable.