6. Use the Ratio Test:
\[
\lim_{n \to \infty} \frac{|x - x_0|^{n+1}}{n+1} \cdot \frac{n}{|x - x_0|^n} = |x - x_0| \lim_{n \to \infty} \left( \frac{n}{n+1} \right) = |x - x_0|
\]

The series converges absolutely if $|x - x_0| < 1$, and diverges if $|x - x_0| > 1$, so the radius is 1.

8. Use the Ratio Test:
\[
\lim_{n \to \infty} \frac{(n+1)!|x|^{n+1}}{(n+1)^{n+1} |x|^n n!} = |x| \lim_{n \to \infty} \left( \frac{n}{n+1} \right)^n
\]

Do you recall the technique where we exponentiate to use L'Hospital’s rule?
\[
\left( \frac{n}{n+1} \right)^n = e^{n \ln \left( \frac{n}{n+1} \right)}
\]

so now we take the limit of the exponent:
\[
\lim_{n \to \infty} n \ln \left( \frac{n}{n+1} \right) = \lim_{n \to \infty} \frac{\ln \left( \frac{n}{n+1} \right)}{\frac{1}{n}}
\]

which is of the form 0/0. Continue with L'Hospital:
\[
\lim_{n \to \infty} \ln \left( \frac{n}{n+1} \right) = \lim_{n \to \infty} \frac{-\frac{1}{n}}{-\frac{1}{n^2}} = \lim_{n \to \infty} \frac{1}{n(n+1)} \cdot -\frac{n^2}{1} = \lim_{n \to \infty} \frac{-n}{n+1} = -1
\]

Therefore,
\[
\lim_{n \to \infty} \left( \frac{n}{n+1} \right)^n = \lim_{n \to \infty} e^{n \ln \left( \frac{n}{n+1} \right)} = e^{-1}
\]

And the ratio test:
\[
\frac{|x|}{e} < 1 \implies |x| < e
\]

12. Actually, this is kind of a “trick question”, although the usual procedure still works:
\[
f(x) = x^2 \implies f(-1) = 1
\]
\[
f'(x) = 2x \implies f'(-1) = -2
\]
\[
f''(x) = 2 \implies f''(-1) = 2
\]
Therefore,
\[ x^2 = 1 - 2(x + 1) + \frac{2}{2!}(x + 1)^2 = 1 - 2(x + 1) + (x + 1)^2 \]
(Notice that if you expand and simplify this, you get \( x^2 \) back.)

This is not an infinite series; no matter what \( x \) is, you can always add those three terms together: The radius of convergence is \( \infty \).

14. At issue here is to find a pattern in the derivatives, so we can write the general form for the \( n^{th} \) derivative.

\[
\begin{align*}
    n = 0 & \quad f(x) = (1 + x)^{-1} & \quad f(0) = 1 \\
    n = 1 & \quad f'(x) = -(1 + x)^{-2} & \quad f'(0) = -1 \\
    n = 2 & \quad f''(x) = (-1)(-2)(1 + x)^{-3} & \quad f''(0) = 2 \\
    n = 3 & \quad f'''(x) = (-1)(-2)(-3)(1 + x)^{-4} & \quad f'''(0) = -3!
\end{align*}
\]

From this we see that:
\[ f^{(n)}(0) = (-1)^n n! \]

The Taylor series (actually, the Maclaurin series) is:
\[
\frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n n! - x^n = \sum_{n=0}^{\infty} (-x)^n
\]
and this converges if \(|x| < 1\) (its an alternating geometric series).

Alternatively, we could see this directly using the sum of the geometric series:
\[
\sum_{n=0}^{\infty} (-x)^n = \frac{1}{1 - (-x)} = \frac{1}{1 + x}
\]

18. Given that
\[ y = \sum_{n=0}^{\infty} a_n x^n \]

Compute \( y' \) and \( y'' \) by writing out the first four terms of each to get the general term. Show that, if \( y'' = y \), then the coefficients \( a_0 \) and \( a_1 \) are arbitrary, and show the given recursion relation.

\[
\begin{align*}
    y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots = \sum_{n=0}^{\infty} a_n x^n \\
    y' &= a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots = \sum_{n=0}^{\infty} (n + 1)a_{n+1} x^n \\
    y'' &= 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + 5 \cdot 4a_5 x^3 + \ldots = \sum_{n=0}^{\infty} (n + 2)(n + 1)a_{n+2} x^n
\end{align*}
\]
If \( y'' = y \), then the coefficients must match up, power by power:

\[
a_0 = 2a_2 \quad a_1 = 6a_3 \quad a_2 = 12a_4 \quad \ldots \quad a_n = (n + 2)(n + 1)a_{n+2}
\]

Problems 19-23 are some symbolic manipulation problems.

19. Rewrite the left side equation so that the powers of \( x \) match up.

20. Much the same. In this problem, we see that the first sum starts with a constant term, the second sum starts with \( x^1 \), and so does the sum on the left. Therefore, we would rewrite each sum to start with \( x^1 \) power:

\[
\sum_{k=1}^{\infty} a_{k+1} x^k = a_1 + \sum_{n=1}^{\infty} a_{n+1} x^n
\]

\[
\sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{n=1}^{\infty} a_{n-1} x^n
\]

Now each sum begins with the same power of \( x \),

\[
\sum_{k=1}^{\infty} a_{k+1} x^k + \sum_{k=0}^{\infty} a_k x^{k+1} = a_1 + \sum_{n=1}^{\infty} a_{n+1} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = a_1 + \sum_{n=0}^{\infty} (a_{n+1} + a_{n-1}) x^n
\]

21. You may use a different symbol for the summation index if you like (it is a dummy variable):

\[
\sum_{n=2}^{\infty} n(n - 1)a_n x^{n-2}
\]

We would like this to be indexed using \( x^k \), \( k = 0, 1, 2, \ldots \) This means that \( k = n - 2 \) or \( n = k + 2 \). Making the substitutions in each term,

\[
\sum_{n=2}^{\infty} n(n - 1)a_n x^{n-2} = \sum_{k=0}^{\infty} (k + 2)(k + 1)a_{k+2} x^k
\]

22. In this case, the powers begin with \( x^2 \), so we let \( k = n + 2 \) or \( n = k - 2 \), with \( k = 2, 3, 4, \ldots \):

\[
\sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{k=2}^{\infty} a_{k-2} x^k
\]

23. Take care of the product with \( x \) first,

\[
x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{k=0}^{\infty} a_k x^k = \sum_{n=1}^{\infty} n a_n x^n + \sum_{k=0}^{\infty} a_k x^k
\]

The first sum could begin with zero- It would make the first term of the sum zero. Therefore,

\[
\sum_{n=0}^{\infty} n a_n x^n + \sum_{k=0}^{\infty} a_k x^k = \sum_{n=1}^{\infty} (n + 1) a_n x^n
\]