2. Solve \( y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi), \) \( y(0) = 0 \) and \( y'(0) = 0. \)

After taking the Laplace transform, we solve for \( Y(s): \)

\[
Y = \frac{e^{-\pi s} - e^{-2\pi s}}{s^2 + 4}
\]

Think of this as:

\[
(e^{-\pi s} - e^{-2\pi s}) H(s)
\]

so that once we find \( h(t) \), the inverse transform will be:

\[
u_\pi(t)h(t - \pi) - u_{2\pi}(t)h(t - 2\pi)
\]

In this case,

\[
H(s) = \frac{1}{s^2 + 4} \quad \Rightarrow \quad h(t) = \frac{1}{2} \sin(2t)
\]

The solution to the IVP can be simplified since \( h \) is periodic with period \( \pi \):

\[
y(t) = u_\pi(t)h(t - \pi) - u_{2\pi}(t)h(t - 2\pi) = h(t)(u_\pi(t) - u_{2\pi}(t))
\]

Therefore, the solution can be written as:

\[
y(t) = \begin{cases} 
\frac{1}{2} \sin(2t) & \text{if } \pi \leq t < 2\pi \\
0 & \text{elsewhere}
\end{cases}
\]

3. \( y'' + 3y' + 2y = \delta(t - 5) + u_{10}(t), \) \( y(0) = 0, \) \( y'(0) = 1/4. \)

Sometimes it is useful to think about what the ODE is before solving it. In this case,

If this represented the model of a mass-spring system, notice that there are three distinct phases of motion- The first begins at time 0, when we have no forcing, but an initial velocity of 1/2. If left alone, the homogeneous solution would die off quickly. However, at \( t = 5 \), the system is given a unit impulse, which starts the system off again (although with a larger velocity that the initial velocity). Again, if left alone, the system would quickly go back to equilibrium. Finally, at time 10, we start a constant force of 1, and continue that through time- We expect our solution to become constant as well (since the homogeneous part of the solution will die off).

Now we’ll solve it algebraically and plot the result in Wolfram Alpha:

\[
(s^2 + 3s + 2)Y = e^{-5s} + \frac{e^{-10s}}{s} + \frac{1}{2}
\]

\[
Y(s) = \frac{1}{s^2 + 3s + 2} \left( e^{-5s} + \frac{1}{2} \right) + \frac{e^{-10s}}{s(s^2 + 3s + 2)}
\]

We have two sets of partial fractions to compute:

\[
\frac{1}{s^2 + 3s + 2} = -\frac{1}{s + 2} + \frac{1}{s + 1}
\]
\[
\frac{1}{s(s^2 + 3s + 2)} = \frac{1}{2s} + \frac{1}{2s + 2} - \frac{1}{s + 1}
\]

Therefore, the inverse Laplace transform gives:

\[
y(t) = \frac{1}{2} \left(-e^{-2t} + e^{-t}\right) + u_5(t) \left(e^{-(t-5)} - e^{-2(t-5)}\right) + u_{10}(t) \left(\frac{1}{2} + \frac{1}{2}e^{-2(t-10)} - e^{-(t-10)}\right)
\]

The solution corresponds to what we had expected. In Wolfram Alpha, we can plot the solution:

```wolfram
solve y''+3y'+2y=Dirac(t-5)+Heaviside(t-10), with y(0)=0, y'(0)=1/2
```

5. The partial fractions here are a little heavy. Here’s how I might do them:

\[
\frac{1}{(s^2 + 1)(s^2 + 2s + 3)} = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 2s + 3}
\]

so that

\[
1 = (As + B)(s^2 + 2s + 3) + (Cs + D)(s^2 + 1)
\]

This leads us to the system of equations:

<table>
<thead>
<tr>
<th>Term</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(s^3) terms</td>
<td>(0 = A + C)</td>
</tr>
<tr>
<td>(s^2) terms</td>
<td>(0 = B + 2A + D)</td>
</tr>
<tr>
<td>(s) terms</td>
<td>(0 = 3A + 2B + C)</td>
</tr>
<tr>
<td>Constants</td>
<td>(1 = 3B + D)</td>
</tr>
</tbody>
</table>

Using the second and fourth equations, we might get a nice substitution:

\[
\begin{align*}
-B - D &= 2A \\
3B + D &= 1 \\
\frac{2B}{1 + 2A} &= B = \frac{1}{2} + A
\end{align*}
\]

And with Equation 1, \(B = \frac{1}{2} - C\). Put these into Equation 3 and we can solve for \(C\):

\[
0 = 3(-C) + 2(1/2 - C) + C \quad \Rightarrow \quad C = \frac{1}{4}
\]

From which we now have \(A = -1/4\), \(B = 1/4\), \(C = 1/4\) and \(D = 1/4\).

7. The tricky part here is computing the Laplace transform of \(\delta(t - c)f(t)\):

\[
\mathcal{L}(\delta(t - c)f(t)) = \int_0^\infty e^{-st}\delta(t - c)f(t)\,dt = e^{-sc}f(c)
\]

where we note that \(f(c)\) is a constant (evaluate \(f\) at \(c\)). In this particular case,

\[
\mathcal{L}(\delta(t - 2\pi)\cos(t)) = e^{-2\pi s} \cdot 1
\]
14. It may be easiest to do this generally, then look at what happens for specific values of \( \gamma \):

\[
y'' + \gamma y' + y = \delta(t - 1) \quad y(0) = y'(0) = 0
\]

Take the Laplace transform of both sides and solve for \( Y(s) \):

\[
Y = \frac{e^{-s}}{s^2 + \gamma s + 1}
\]

Our choice of table entry for inversion depends on whether or not the denominator is irreducible. We can tell by completing the square:

\[
s^2 + \gamma s + 1 = \left(s + \frac{\gamma}{2}\right) + \left(1 - \frac{\gamma^2}{4}\right) = \left(s + \frac{\gamma}{2}\right)^2 + \left(\frac{\sqrt{4 - \gamma^2}}{2}\right)^2
\]

In the cases we are asked to consider, \( \gamma = 1/2, 1/4 \) and 0, the denominator is irreducible. Now invert the transform: Given

\[
H(s) = \frac{2}{\sqrt{4 - \gamma^2}} \frac{\frac{\sqrt{4 - \gamma^2}}{2}}{(s + \frac{\gamma}{2})^2 + \left(\frac{\sqrt{4 - \gamma^2}}{2}\right)^2}
\]

then

\[
h(t) = \frac{2}{\sqrt{4 - \gamma^2}} e^{-(\gamma/2)t} \sin \left(\frac{\sqrt{4 - \gamma^2}}{2} t\right)
\]

The overall solution is then \( u_1(t)h(t - 1) \).

For parts (b) and (c), we are meant to use the computer to solve for the maximum. We can answer part (d): If \( \gamma = 0 \), then solution simplifies to

\[
h(t) = \sin(t)
\]

so that the maximum of \( h(t) \) occurs at \( t = \pi/2 \) (so the maximum of \( h(t - 1) \) occurs at \( t = 1 + \pi/2 \).

15. The solution to this one is almost identical to the previous problem, except we multiply by \( k \):

\[
y(t) = ku_1(t)h(t - 1)
\]

where \( h(t) \) was found in #14. The remaining problems are meant to be done on a computer.

16. Omit this problem.

17-19. In these problems, we work with using the sum. Try to think about how the sum of the impulses will effect your solution- The homogeneous solution is simply a sum of \( \sin(t) \) and \( \cos(t) \), so that the homogeneous part of the solution has a period of \( 2\pi \).
17. In this problem, the first impulse occurs at \( t = \pi \), and so that will start a sine function:

\[
(s^1 + 1)Y(s) = e^{-\pi s} \Rightarrow Y(s) = \frac{e^{-\pi s}}{s^2 + 1} \Rightarrow y(t) = u_\pi(t) \sin(t - \pi)
\]

At \( t = 2\pi \) comes our next unit impulse. Note (from a sketch of \( y(t) \)) that \( y'(2\pi) = -1 \), so the impulses will cancel each other out. Algebraically,

\[
y(t) = u_\pi(t) \sin(t - \pi) + u_{2\pi}(t) \sin(t - 2\pi) = u_\pi(t)(-\sin(t)) + u_{2\pi}(t) \sin(t)
\]

Writing it piecewise,

\[
y(t) = \begin{cases} 
0 & \text{if } 0 \leq t < \pi \\
-\sin(t) & \text{if } \pi \leq t < 2\pi \\
0 & \text{if } t \geq 2\pi 
\end{cases}
\]

Now when \( \delta(t-3\pi) \) comes along, it starts the same motion as before (then \( \delta(t-4\pi) \) turns it off again, then \( \delta(t-5\pi) \) starts it up again, etc.). Therefore, the solution (in piecewise form) is the following- After time \( 20\pi \), the solution will be zero following the pattern:

\[
y(t) = \begin{cases} 
0 & \text{if } 0 \leq t < \pi \\
-\sin(t) & \text{if } \pi \leq t < 2\pi \\
0 & \text{if } 2\pi \leq t < 3\pi \\
-\sin(t) & \text{if } 3\pi \leq t < 4\pi \\
\vdots & \vdots \\
-\sin(t) & \text{if } 19\pi \leq t < 20\pi \\
0 & \text{if } t \geq 20\pi 
\end{cases}
\]

18. In Exercise 18, we start the same way- at \( t = \pi \) we impart a unit impulse, and that starts a sine function going (that is, \( \sin(t - \pi) = -\sin(t) \)).

After \( \pi \) units of time (\( t = 2\pi \)), the curve has a velocity of \(-1\), and we impart an additional unit impulse in the negative direction (that will make the amplitude increase by 1).

Similarly, at \( t = 3\pi \), the curve now has a velocity of 2, and we will impart an additional unit impulse in the positive direction (so the amplitude increases to 3).

The same thing happens at \( t = 4\pi \), \( t = 5\pi \), etc. Therefore, the sine function will continue to grow 1 unit in amplitude for every \( \pi \) units in time until we get to \( 20\pi \). After that, the solution will have an amplitude of 20.

Let’s see if we can show that algebraically: We know that

\[
\sin(t - k\pi) = -\sin(t) \quad k = 1, 3, 5, 7, \ldots
\]

and

\[
\sin(t - k\pi) = \sin(t) \quad k = 2, 4, 6, 8, \ldots
\]
Therefore,

\[
y(t) = \sum_{k=1}^{20} (-1)^{k+1} u_{k\pi}(t) \sin(t - k\pi) = \begin{cases} 
0 & \text{if } 0 \leq t \leq \pi \\
-\sin(t) & \text{if } \pi \leq t < 2\pi \\
-2\sin(t) & \text{if } 2\pi \leq t < 3\pi \\
-3\sin(t) & \text{if } 3\pi \leq t < 4\pi \\
\vdots & \\
-19\sin(t) & \text{if } 19\pi \leq t < 20\pi \\
-20\sin(t) & \text{if } t \geq 20\pi 
\end{cases}
\]

19. This one is more complex since the “hits” don’t occur at the end of a period (rather they occur in the middle of a period).

We can analyze this easiest by writing the solution piecewise using the following substitutions (do them graphically if you’re not sure):

\[
\begin{align*}
\sin(t - \pi/2) &= \cos(t) \\
\sin(t - \pi) &= -\sin(t) \\
\sin(t - 3\pi/2) &= -\cos(t) \\
\sin(t - 2\pi) &= \sin(t)
\end{align*}
\]

Therefore, we end up with a function that is \(2\pi\) periodic:

\[
y(t) = \begin{cases} 
0 & \text{if } 0 \leq t < \pi/2 \\
\cos(t) & \text{if } \pi/2 \leq t < \pi \\
\cos(t) - \sin(t) & \text{if } \pi \leq t < 3\pi/2 \\
-\sin(t) & \text{if } 3\pi/2 \leq t < 2\pi \\
0 & \text{if } 2\pi \leq t < 3\pi/2 \\
\vdots & \\
0 & \text{if } t \geq 20\pi
\end{cases}
\]