Selected Solutions: Section 6.1

1. This is piecewise continuous, but not continuous at $t = 1$.

2. Not continuous and not piecewise continuous.

3. Continuous (so also piecewise continuous).

5. Sketched the solution in class- Use a table.

21. Recall that the inverse tangent function has a limit as $t \to \infty$; the function approaches $\pi/2$ (which is a vertical asymptote for the original tangent).

23. Use the test for divergence: If the limit of $f(t)$ (as $t \to \infty$) is not zero, the improper integral diverges.

26. (Done in class) The Gamma Function $\Gamma(p)$

   (a) If $p > 0$, then show $\Gamma(p + 1) = p\Gamma(p)$:

   $$\Gamma(p + 1) = \int_0^\infty e^{-x}x^p \, dx$$

   Integration by parts gives us the answer for $p > 0$. Actually, the following is true for $p > -1$:

   $$\begin{align*}
   &\left. x^p \right|_{-e^{-x}} - \frac{x^p}{px^{p-1}} \to -x^pe^{-x}\bigg|_0^\infty + p \int_0^\infty e^{-x}x^{p-1} \, dx
   
   
   \end{align*}$$

   The quantity $-x^pe^{-x}$ goes to zero as $x \to \infty$ for any $p$. However, if $p$ is negative we have to be careful about $x^p$ as $x \to 0$. If we restrict $p > 0$, then $x^pe^{-x} = 0$ at zero, and we get:

   $$\Gamma(p + 1) = \int_0^\infty e^{-x}x^p \, dx = p\int_0^\infty e^{-x}x^{p-1} \, dx = p\Gamma(p)$$

   (b) Show that $\Gamma(1) = 1$. We can do this directly by taking $p = 0$:

   $$\int_0^\infty e^{-x} \, dx = -e^{-x}\bigg|_0^\infty = 0 - (-1) = 1$$

   (c) If $p$ is a positive integer, show that $\Gamma(n + 1) = n!$.

   We can show this by induction. We note from parts (a) and (b) that:

   $$\Gamma(1) = 1 \quad \Gamma(2) = 1 \cdot \Gamma(1) = 1 \quad \Gamma(3) = 2 \cdot \Gamma(2) = 2 \cdot 1$$

   In this case, we showed that the formula works if $n = 1, 2$ or 3 (not necessary, but it does give you a general idea).
Assume that the formula works for \( n = k \), \( \Gamma(k + 1) = k! \). Show that it works for \( n = k + 1 \). By Part (a),

\[
\Gamma(k + 2) = (k + 1)\Gamma(k + 1)
\]

And by what we assumed, if \( k + 2 \) is a positive integer, then

\[
\Gamma(k + 2) = (k + 1)\Gamma(k + 1) = (k + 1)! = (k + 1)!
\]

Therefore, we have proved by induction that \( \Gamma(n + 1) = n! \).

(d) (This part can be omitted) By repeating the process in (c),

\[
\Gamma(p + n) = p\Gamma(p + n - 1) = (p + n - 1)(p + n - 2)\Gamma(p + n - 2) = \\
= \ldots = p(p + 1)(p + 2) \cdots (p + n - 1)\Gamma(p)
\]

27. (a) Hint: Let \( x = st \), then do a change of variables.

(b) Straightforward- Use the result of 26.

(c) This is an interesting problem, but may be omitted. Assuming the formulas given in the text,

\[
\mathcal{L}(t^{-1/2}) = \int_0^\infty e^{-st} \frac{1}{\sqrt{t}} \, dt
\]

Looking at what we want, we’ll try setting \( x^2 = st \) and perform a substitution. Finding \( dx \) and \( dt \), we get:

\[
2x \, dx = s \, dt \quad \Rightarrow \quad 2\sqrt{s} \, dx = s \, dt \quad \Rightarrow \quad \frac{2}{\sqrt{s}} \, dx = \frac{1}{\sqrt{t}} \, dt
\]

which is what we needed to get the expression in the text:

\[
\mathcal{L}(t^{-1/2}) = \frac{2}{\sqrt{s}} \int_0^\infty e^{-x^2} \, dx = \sqrt{\frac{\pi}{s}}
\]

(d) Finally, we’ll use the result from 26: \( \Gamma(3/2) = \frac{1}{2} \Gamma(1/2) \) to compute this:

\[
\mathcal{L}(t^{1/2}) = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}
\]