Help on section 6.2

The main ideas:

- Take the Laplace transform of a DE.
- Solve it for \( Y(s) \) (the transform of \( y(t) \)).
- Invert the transform- This may involve partial fractions and/or completing the square.

The first set of exercises focuses on this last part, where partial fractions and/or completing the square will be performed in order to use the table to do the inverse transform.

2. Table entry 11.

4. Use Partial Fractions:

\[
\frac{3s}{s^2 - s - 6} = \frac{A}{s + 2} + \frac{B}{s - 3}
\]

so that

\( A(s - 3) + B(s + 2) = 3s \)

If \( s = 3 \), then \( B = 9/5 \). If \( s = -2 \), then \( A = 6/5 \). Now,

\[
\mathcal{L}^{-1}\left(\frac{6}{5} \frac{1}{s + 2} + \frac{9}{5} \frac{1}{s - 3}\right) = \frac{6}{5} \mathcal{L}^{-1}\left(\frac{1}{s + 2}\right) + \frac{9}{5} \mathcal{L}^{-1}\left(\frac{1}{s - 3}\right) = \frac{6}{5}e^{-2t} + \frac{9}{5}e^{3t}
\]

You may use Wolfram Alpha to check your answer:

inverse laplace transform of \( 3s/(s^2 - 2s - 6) \)

8. Use Partial fractions:

\[
\frac{8s^2 - 4s + 12}{s(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4}
\]

Therefore,

\( 8s^2 - 4s + 12 = A(s^2 + 4) + (Bs + C)s \) \( \Rightarrow \) \( 8s^2 - 4s + 12 = (A + B)s^2 + Cs + 4A \)

Therefore,

\[
\begin{align*}
A + B &= 4 \\
C &= -4 \\
4A &= 12
\end{align*}
\]

\( \Rightarrow \) \( A = 3, C = -4, B = 1 \)

And we get:

\[
\frac{8s^2 - 4s + 12}{s(s^2 + 4)} = \frac{3}{s} + \frac{5s - 4}{s^2 + 4}
\]
For the inverse Laplace transform, we look at this as:

\[ \frac{3}{s} + \frac{5}{s^2 + 4} - \frac{2}{s^2 + 4} \]

so that our final answer is:

\[ 3 + 5\cos(2t) - 2\sin(2t) \]

In Wolfram Alpha, you can ask for just the partial fraction decomposition:

\[ \text{partial fraction} \left( \frac{8s^2-4s+12}{s(s^2+4)} \right) \]

10. Complete the square in the denominator:

\[ \frac{2s-3}{s^2+2s+1+9} = \frac{2(s+1)-5}{(s+1)^2+9} = \frac{2}{(s+1)^2+3^2} - \frac{5}{3(s+1)^2+3^2} \]

12. Take the Laplace transform of both sides:

\[ s^2Y - sy(0) - y'(0) + 3(sY - y(0)) + 2Y = 0 \]

Apply initial conditions and solve for \( Y(s) \):

\[ Y(s) = \frac{s+3}{s^2+3s+2} = \frac{2}{s+1} + \frac{1}{s+2} \]

so that

\[ y(t) = 2e^{-t} - e^{-2t} \]

14. Same idea:

\[ s^2Y - sy(0) - y'(0) - 4(sY - y(0)) + 4Y = 0 \]

Apply initial conditions and solve for \( Y(s) \):

\[ Y(s) = \frac{s-3}{(s^2-4s+4)} = \frac{(s-3)}{(s-2)} = \frac{(s-2)-1}{(s-2)^2} = \frac{1}{s-2} - \frac{1}{(s-2)^2} \]

Use table entries 2 and 11:

\[ y(t) = e^{2t} - te^{2t} \]

20. Continuing in the same fashion as before, you should find that

\[ Y(s) = \frac{s}{(s^2 + \omega^2)(s^2 + 4)} + \frac{s}{s^2 + \omega^2} \]

Use Partial Fractions on the first term:

\[ \frac{s}{(s^2 + \omega^2)(s^2 + 4)} = \frac{As + B}{s^2 + \omega^2} + \frac{Cs + D}{s^2 + 4} \]
so that (multiply it out and equate coefficients):

\[
\begin{align*}
A + C &= 0 \\
B + D &= 0 \\
4A + C\omega^2 &= 1 \\
4B + D\omega^2 &= 0
\end{align*}
\Rightarrow A = \frac{1}{4 - \omega^2}, C = -A, B = 0, D = 0
\]

22. Same idea as before.

\[
Y(s) = \frac{1}{(s + 1)(s^2 - 2s + 2)} + \frac{1}{s^2 - 2s + 2}
\]

Using partial fractions again, get that:

\[
Y(s) = \frac{1/5}{s + 1} - \frac{1}{5} \cdot \frac{s - 3}{s^2 - 2s + 2} + \frac{1}{s^2 - 2s + 2}
\]

Now complete the square in the denominator of the last two terms.

24. For the Laplace transform of the right hand side of the given differential equation, define \(f(t)\) as the piecewise defined function. Then:

\[
\int_0^\infty e^{-st} f(t) \, dt = \int_0^\pi e^{-st} \, dt + \int_\pi^\infty 0 \, dt = -\frac{e^{-st}}{s}\bigg|_0^\pi = \frac{1 - e^{-\pi s}}{s}
\]

Therefore, the Laplace transform will be:

\[
Y(s) = \frac{1 - e^{-\pi s}}{s} + \frac{s}{s^2 + 4}
\]

28.

\[
F'(s) = \frac{d}{ds} \left( \int_0^\infty e^{-st} f(t) \, dt \right) = \int_0^\infty \frac{d}{ds} \left( e^{-st} \right) f(t) \, dt = \int_0^\infty -te^{-st} f(t) \, dt = \mathcal{L}(-tf(t))
\]

29. Use Exercise 28, which in this case is

\[
-\mathcal{L}(-te^{at}) = -F'(s)
\]

where \(F(s) = \frac{1}{s-a}\). Therefore,

\[
-\mathcal{L}(-te^{at}) = \frac{1}{(s-a)^2}
\]

As in table entry 11 with \(n = 1\).
30. Same idea as Exercise 29, except we differentiate twice (a little messy) to get:

\[
\mathcal{L}(t^2 \sin(bt)) = \frac{d^2}{ds^2} \left( \frac{b}{s^2 + b^2} \right) = \frac{d}{ds} \left( (-2bs)(s^2 + b^2)^{-2} \right) = \frac{6bs^2 - 2b^3}{(s^2 + b^2)^3}
\]

33. Differentiate table entry 9 with respect to \( s \), then multiply by \(-1\).

37. This one is a bit tricky (A Challenge Problem!)

\[
\mathcal{L}(g(t)) = \int_0^\infty e^{-st} \left[ \int_0^t f(w) \, dw \right] \, dt = \int_0^\infty \int_0^t e^{-st} f(w) \, dw \, dt
\]

In the \((t, w)\) plane, we have \(0 \leq w \leq t \) and \(0 \leq t \leq \infty\). If we reverse the order of integration then \(t \geq w\) and \(0 \leq w \leq \infty\) (draw it in the \((t, w)\) plane). Therefore,

\[
\int_0^\infty f(w) \left[ \int_w^\infty e^{-st} \right] \, dt \, dw = \int_0^\infty f(w) \frac{e^{-sw}}{s} \, dw
\]

which is what we wanted: \( \frac{1}{s} \mathcal{L}(f(t)) \).