Homework from Section 2.3

1. Exercise 2.3.2

(a) We consider the BVP: $\phi'' + \lambda \phi = 0$, with $\phi(0) = 0$ and $\phi(\pi) = 0$.

*Side Remark:* In this exercise, we should practice making the three cases for $\lambda$.

**SOLUTION:** The characteristic equation is $r^2 + \lambda = 0$, or $r = \pm \sqrt{-\lambda}$.

- If $\lambda = 0$, we have one real root ($r = 0$) and the solution is $C_1 x + C_2$. Applying the boundary conditions, we get $C_1 = C_2 = 0$, and we get only the trivial solution.
- If $\lambda < 0$, then we have two real solutions and we can write the general solution as the following (getting practice with sinh and cosh):
  
  $$C_1 \cosh(\sqrt{-\lambda} x) + C_2 \sinh(\sqrt{-\lambda} x)$$

  Using the boundary conditions,
  
  $$\phi(0) = 0 \implies C_1 + 0 = 0$$
  $$\phi(\pi) = 0 \implies C_2 \sinh(\sqrt{-\lambda} \pi) = 0$$

  Since $\lambda \neq 0$, this is true only if $C_2 = 0$. In this case, we also get only the trivial solution.
- Last case is $\lambda > 0$, in which case we get two complex solutions to the characteristic equation, and the general solution is:
  
  $$\phi(x) = C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x)$$

  Applying the boundary conditions, $\phi(0) = 0$ implies $C_1 = 0$, so the remaining condition gives us:
  
  $$C_2 \sin(\sqrt{\lambda} \pi) = 0$$

  Either $C_2 = 0$ (which leads us back to the trivial solution), or the sine evaluates to zero. The sine evaluates to zero if:
  
  $$\sqrt{\lambda} \pi = n \pi \quad \text{for } n = 1, 2, 3, \cdots \implies \lambda = n^2 \quad \text{for } n = 1, 2, 3, \cdots$$

  Our solutions are therefore of the form:
  
  $$\phi_n = C_n \sin(n x) \quad \text{for } n = 1, 2, 3, \cdots$$

2. Exercise 2.3.3(b,c)

In Section 2.3.3, we are given the heat equation with zero boundary conditions and varying initial functions. It is fine if you are already very familiar with separation of variables to start with the general form of the solution:

$$u(x, t) = \sum_{n=1}^{\infty} B_n e^{-k(n\pi/L)^2 t} \sin \left( \frac{n\pi}{L} x \right)$$
where

\[ B_n = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{n\pi x}{L} \right) \]

**HOWEVER**, if you’re not exactly sure where any of those terms come from, you should be able to get the general solution using separation of variables and the integral formulas for the sine and cosine.

(b) In this case, the coefficients \( B_n \) are going to be zero except for \( n = 1 \) and \( n = 3 \) (by the orthogonality of the sines):

\[ B_1 = \frac{2}{L} \int_0^L 3 \sin^2 \left( \frac{\pi x}{L} \right) dx = \frac{3}{2} \frac{L^2}{2} = 3 \]

And similarly, \( B_3 = -1 \). Therefore, the full solution is:

\[ u(x, t) = 3e^{-k(\pi/L)^2 t} \sin \left( \frac{\pi x}{L} \right) - e^{-k(3\pi/L)^2 t} \sin \left( \frac{3\pi x}{L} \right) \]

(c) We already know the formula for the coefficients:

\[ u(x, t) = \sum_{n=1}^{\infty} B_n e^{-k(n\pi/L)^2 t} \sin \left( \frac{n\pi x}{L} \right) \]

where

\[ B_n = \frac{2}{L} \int_0^L 2 \cos \left( \frac{3\pi x}{L} \right) \sin \left( \frac{n\pi x}{L} \right) \]

(We’ll be computing these in Chapter 3, that’s why the little footnote was there).

3. **Exercise 2.3.4** Here we have our friend the heat equation,

\[ u_t = k u_{xx} \]

with zero boundary conditions and initial temperature profile \( f(x) \).

(a) Let’s compute the total heat energy in the rod as a function of time.

In this case, we are meant to use the general formula for the solution:

\[ E(t) = \int_0^L c \rho u(x, t) A \, dx = c \rho A \sum_{n=1}^{\infty} B_n e^{-k(n\pi/L)^2 t} \left[ \int_0^L \sin \left( \frac{n\pi x}{L} \right) \, dx \right] \]

The quantity in the square brackets is:

\[ \frac{-L}{n\pi} \cos \left( \frac{n\pi x}{L} \right) \bigg|_0^L = \frac{-L}{n\pi} (\cos(n\pi) - 1) \]

Now, we should simplify further: \( \cos(n\pi) = (-1)^n \), so that

\[ \int_0^L \sin \left( \frac{n\pi x}{L} \right) \, dx = \frac{L(1 + (-1)^{n+1})}{n\pi} \]

Substitute that expression back into the square brackets for the energy, and we note that this is a function of time.
(b) What is the flow of heat energy out of the rod at $x = 0$? At $x = L$?

SOLUTION: With the general solution, we can compute $u_x(0, t)$ and $u_x(L, t)$, since the flux is $\phi(0, t) = -K_0 u_x(0, t)$ and $\phi(L, t) = -K_0 u_x(L, t)$.

$$
\sum_{n=1}^{\infty} B_n e^{-k(n\pi/L)^2 t} \sin \left( \frac{n\pi}{L} x \right)
$$

so that:

$$
u_x(x, t) = \sum_{n=1}^{\infty} B_n e^{-k(n\pi/L)^2 t} \frac{n\pi}{L} \cos \left( \frac{n\pi}{L} x \right)
$$

so that

$$
u_x(0, t) = \frac{\pi}{L} \sum_{n=1}^{\infty} n B_n e^{-k(n\pi/L)^2 t}
$$

$$
u_x(L, t) = \frac{\pi}{L} \sum_{n=1}^{\infty} (-1)^n n B_n e^{-k(n\pi/L)^2 t}
$$

(c) What relationship should exist between parts (a) and (b)? (See back of text). We don’t have to actually show this relationship here.