Homework from Section 2.5

The assignment was 1(a,f), 2, 3, 5(a), 7(a), 10.

2.5.1(a)  Given the boundary conditions, we can translate them using separation of variables to:

\[ X'(0) = 0 \quad X'(L) = 0 \quad Y(0) = 0 \quad Y(H) \neq 0 \]

Separating variables, we have the two ODEs:

\[ X'' + \lambda X = 0 \quad Y'' - \lambda Y = 0 \]

Now, solve \( X'' + \lambda X = 0 \) by solving the characteristic equation, \( r = \pm \sqrt{-\lambda} \)

- If \( \lambda = 0 \), then \( X(x) = C_1 x + C_2 \). Applying the boundary values, we see that \( C_1 = 0 \), and \( X_0 = 1 \) (think of \( X_0 \) as an arbitrary constant times the function 1).
- If \( \lambda < 0 \):
  \[
  X(x) = C_1 \cosh(\sqrt{-\lambda}x) + C_2 \sinh(\sqrt{-\lambda}x)
  \]
  With \( X'(0) = 0 \), then \( C_2 = 0 \). For the other boundary condition,
  \[
  X'(L) = 0 \quad \Rightarrow \quad \sqrt{-\lambda}C_1 \sinh(\sqrt{-\lambda}L) = 0
  \]
  The hyperbolic sine is zero only at the origin (and \( \lambda, L \) are not zero), so therefore, \( C_1 \) must be zero and we're left with the trivial solution again.
- Final case: \( \lambda > 0 \): In this situation, the characteristic equation has complex roots \( r = \pm \sqrt{\lambda}i \) so the general solution is:
  \[
  X = A \cos(\sqrt{\lambda}x) + B \sin(\sqrt{\lambda}x)
  \]
  With \( X'(0) = 0 \), we get \( B = 0 \), and with \( X'(L) = 0 \), we get the eigenvalues
  \[
  \sqrt{\lambda}L = n\pi \quad \text{for} \quad n = 1, 2, 3, \ldots
  \]
  Therefore, \( X_0 = 1 \), and
  \[
  X_n(x) = A_n \cos\left(\frac{n\pi}{L} x\right) \quad \text{with} \quad \lambda = \left(\frac{n\pi}{L}\right)^2
  \]
  Now, going to solve for \( Y \):
  \[
  Y'' - \lambda Y = 0
  \]
  with \( \lambda \) as found, we have two cases to consider:
  - \( \lambda = 0 \) for \( n = 0 \): \( Y(y) = C_1 y + C_2 \); with \( Y(0) = 0 \), then \( C_2 = 0 \), and \( Y_0 = y \).
  - \( \lambda > 0 \):
    \[
    Y(y) = C_n \cosh\left(\frac{n\pi}{L} y\right) + D_n \sinh\left(\frac{n\pi}{L} y\right)
    \]
With \( Y(0) = 0 \), \( C_n = 0 \), and we can write down the expansion:

\[
  u(x, y) = A_0 X_0 Y_0 + \sum_{n=1}^{\infty} X_n Y_n
\]

which gives us:

\[
  u(x, y) = A_0 y + \sum_{n=1}^{\infty} \sinh \left( \frac{n\pi}{L} y \right) \cos \left( \frac{n\pi}{L} x \right)
\]

To find the coefficients, we use the initial condition:

\[
  u(x, H) = f(x) = A_0 H + \sum_{n=1}^{\infty} \cosh \left( \frac{n\pi}{L} y \right) \cos \left( \frac{n\pi}{L} x \right)
\]  

(1)

Multiply both sides by 1 and integrate \( x \) from 0 to \( L \) to get:

\[
  \int_0^L f(x) \, dx = A_0 HL + 0 + 0 + 0 + \cdots \quad \Rightarrow \quad A_0 = \frac{1}{HL} \int_0^L f(x) \, dx
\]

Going back to Equation 1, find the \( k^{th} \) coefficient by multiplying both sides by \( \cos(k\pi x/L) \), and integrate:

\[
  \int_0^L f(x) \cos \left( \frac{k\pi x}{L} \right) \, dx = 0 + 0 + \cdots + \int_0^L A_k \sinh \left( \frac{k\pi H}{L} \right) \cos^2 \left( \frac{k\pi x}{L} \right) + 0 + 0 + \cdots
\]

Therefore, for \( k = 1, 2, 3, \cdots \), we have:

\[
  A_k = \frac{1}{\sinh \left( \frac{k\pi H}{L} \right)} \frac{2}{L} \int_0^L f(x) \cos \left( \frac{k\pi x}{L} \right) \, dx
\]

2.5.1(f) Looking at the boundary conditions, we’ll now have:

\[
  X(L) = 0 \quad Y''(0) = 0 \quad Y'(H) = 0
\]

so we’ll choose \( \lambda \) so that

\[
  Y'' + \lambda Y = 0 \quad X'' - \lambda X = 0
\]

We already know what’s going to happen, but let’s go through the cases anyway just to be thorough. We’ll solve the DE in \( Y \) first:

- With \( \lambda = 0 \), we have \( Y(y) = C_1 y + C_2 \). Applying the boundary conditions, \( C_1 = 0 \), but we could have a constant solution, \( Y_0 = 1 \) (we’ll multiply by an arbitrary constant later).
• With \( \lambda < 0 \), we have
\[
Y(y) = C_1 \cosh(\sqrt{-\lambda} y) + C_2 \sinh(\sqrt{-\lambda} y)
\]
so that the derivative is
\[
Y'(y) = \sqrt{-\lambda} C_1 \sinh(\sqrt{-\lambda} y) + \sqrt{-\lambda} C_2 \cosh(\sqrt{-\lambda} y)
\]
Therefore, \( Y'(0) = 0 \) implies that \( C_2 = 0 \), and \( Y'(H) = 0 \) implies that
\[
\sqrt{-\lambda} C_1 \sinh(\sqrt{-\lambda} H) = 0
\]
We know that \( \lambda \neq 0 \), so that the hyperbolic sine is not zero. Therefore, \( C_1 = 0 \). In this case, we get only the trivial solution.

• \( \lambda > 0 \). In this case,
\[
Y(y) = C_1 \cos(\sqrt{\lambda} y) + C_2 \sin(\sqrt{\lambda} y)
\]
so that the derivative is
\[
Y'(y) = -\sqrt{\lambda} C_1 \sin(\sqrt{\lambda} y) + \sqrt{\lambda} C_2 \cos(\sqrt{\lambda} y)
\]
Therefore, \( Y'(0) = 0 \) implies that \( C_2 = 0 \), and \( Y'(H) = 0 \) implies that
\[
\sqrt{\lambda} C_1 \sin(\sqrt{\lambda} H) = 0
\]
Now this implies that
\[
\lambda = \left(\frac{n\pi}{H}\right)^2 \quad \text{for} \quad n = 1, 2, 3, \ldots \quad \Rightarrow \quad Y_n(y) = A_n \cos\left(\frac{n\pi}{H} y\right)
\]
Going to the ODE in \( X \), there are two cases for \( \lambda \):

• \( \lambda = 0 \): \( X = C_1 x + C_2 \) and given \( X(0) \neq 0 \) and \( X(L) = 0 \), we’ll take \( X_0 = x - L \).

• \( \lambda > 0 \), so the only solutions in \( X \) we need to consider are the hyperbolic sine and cosine. Since we won’t be evaluating at zero until later, and we will be evaluating at \( L \) now, we can write the solution as:
\[
X(x) = C_1 \cosh\left(\frac{n\pi}{H} (x - L)\right) + C_2 \sin\left(\frac{n\pi}{H} (x - L)\right)
\]
Therefore, the condition \( X(L) = 0 \) will imply that \( C_1 \) is zero, and (leaving the constants with \( Y \)), we can write
\[
X_n(x) = \sinh\left(\frac{n\pi}{H} (x - L)\right)
\]
Now we have the full generic solution:

\[ u(x, y) = A_0(x - L) + \sum_{n=1}^{\infty} A_n \sinh \left( \frac{n\pi}{H} (x - L) \right) \cos \left( \frac{n\pi}{H} y \right) \]

Looking at the initial function of \( y \), we have:

\[ u(0, y) = f(y) = A_0(-L) + \sum_{n=1}^{\infty} A_n \sinh \left( \frac{n\pi}{H} (-L) \right) \cos \left( \frac{n\pi}{H} y \right) = A_0(-L) + \sum_{n=1}^{\infty} \hat{A}_n \cos \left( \frac{n\pi}{H} y \right) \]

We have collected the hyperbolic sine term in with the constant so that the relationships are more clear. That is,

\[ \hat{A}_n = A_n \sinh \left( \frac{n\pi}{H} (-L) \right) \]

Now we compute the terms as usual:

\[ \int_0^H f(y) \, dy = A_0(-L) \int_0^H 1 \, dy + \sum_{n=1}^{\infty} \hat{A}_n \int_0^H \cos \left( \frac{n\pi}{H} y \right) \, dy = A_0(-LH) + 0 + 0 + \cdots \]

Therefore,

\[ A_0 = -\frac{1}{LH} \int_0^H f(y) \, dy \]

To find the \( n^{th} \) constant, we would multiply both sides of the solution by \( \cos \left( \frac{n\pi y}{H} \right) \), then integrate from 0 to \( H \):

\[ \int_0^H f(y) \cos \left( \frac{n\pi y}{H} \right) \, dy = A_0 \int_0^H \cos \left( \frac{n\pi y}{H} \right) \, dy + \hat{A}_1 \int_0^H \cos \left( \frac{n\pi y}{H} \right) \, dy + \cdots + \hat{A}_n \int_0^H \cos^2 \left( \frac{n\pi y}{H} \right) \, dy + \cdots \]

So that we’re just left with:

\[ \hat{A}_n = \frac{2}{H} \int_0^H f(y) \cos \left( \frac{n\pi y}{H} \right) \, dy \]

And, solving for the coefficients without hats:

\[ A_n = \frac{1}{\sinh \left( \frac{n\pi}{H} (-L) \right)} \cdot \frac{2}{H} \int_0^H f(y) \cos \left( \frac{n\pi y}{H} \right) \, dy \]

2.5.2 (a) Since the solution to this equation can be regarded as an equilibrium solution to the heat equation in two dimensions, and since there is no source for heat inside the plate (no \( Q \)), the total heat flowing in must be equal to the total heat flowing out. In this case, the heat flow is zero at each boundary except along \( (x, H) \), \( 0 \leq x \leq L \), and so the total flow through this boundary must be zero:

\[ \int_0^L u_y(x, H) \, dx = \int_0^L f(x) \, dx = 0 \]
(b) Solving the DE, we consider the boundary conditions:

\[ X'(0) = 0 \quad X'(L) = 0 \quad Y'(0) = 0 \quad Y'(H) \neq 0 \]

We have two boundary values for \( X \), and we suspect it to be periodic, so we’ll \( \lambda \) so that:

\[ X'' + \lambda X = 0 \quad Y'' - \lambda Y = 0 \]

Taking care of \( \lambda = 0 \) first,

\[ X(x) = C_1 x + C_2 \quad Y(y) = C_3 y + C_4 \]

Applying the boundary conditions, we see that \( C_1 = C_3 = 0 \), and the constant function solves both. Continuing,

- Can \( \lambda < 0 \)? In that case,

\[ X(x) = C_1 \cosh(\sqrt{-\lambda}x) + C_2 \sinh(\sqrt{-\lambda}x) \]

and

\[ X'(x) = \sqrt{-\lambda}C_1 \sinh(\sqrt{-\lambda}x) + \sqrt{-\lambda}C_2 \cosh(\sqrt{-\lambda}x) \]

Therefore, \( X'(0) = 0 \) forces \( C_2 = 0 \), and \( X'(L) = 0 \) means:

\[ C_1 \sinh(\sqrt{-\lambda}L) = 0 \]

which forces \( C_1 = 0 \).

- Now back to our standard case, \( \lambda > 0 \). In that case,

\[ X(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x) \]

and

\[ X'(x) = -\sqrt{\lambda}C_1 \sin(\sqrt{\lambda}x) + \sqrt{\lambda}C_2 \cos(\sqrt{\lambda}x) \]

Now, \( X'(0) = 0 \) forces \( C_2 = 0 \), and \( X'(L) = 0 \) gives us:

\[ C_1 \sin(\sqrt{\lambda}L) = 0 \quad \Rightarrow \quad \lambda = \left( \frac{n\pi}{L} \right)^2, \ n = 1, 2, 3, \cdots \]

Now we’re back to the \( Y \)'s. We have only the case that \( \lambda = n^2\pi^2/L^2 \)

\[ Y(y) = C_1 \cosh \left( \frac{n\pi y}{L} \right) + C_2 \sinh \left( \frac{n\pi y}{L} \right) \]

Now, \( Y'(0) = 0 \) implies that \( C_2 = 0 \), so that leaves

\[ Y_n(y) = \cosh \left( \frac{n\pi y}{L} \right) \]
and our full general solution is:

\[ u(x, y) = A_0 + \sum_{n=1}^{\infty} A_n \cosh \left( \frac{n\pi y}{L} \right) \cos \left( \frac{n\pi x}{L} \right) \]

This initial condition is different than before, since \( f(x) \) is the derivative of \( u \).

\[ u_y(x, y) = \sum_{n=1}^{\infty} A_n \frac{n\pi}{L} \sinh \left( \frac{n\pi y}{L} \right) \cos \left( \frac{n\pi x}{L} \right) \]

Therefore,

\[ f(x) = \sum_{n=1}^{\infty} \left[ A_n \frac{n\pi}{L} \sinh \left( \frac{n\pi H}{L} \right) \right] \cos \left( \frac{n\pi x}{L} \right) \]  

(2)

where the quantity in the square brackets is a constant for each \( n \). This constant is found in the usual way- To find the \( m \)th one, multiply both sides by \( \cos \left( \frac{m\pi x}{L} \right) \) and integrate from \( 0 \leq x \leq L \):

\[ \int_0^L f(x) \cos \left( \frac{m\pi x}{L} \right) \, dx = 0 + 0 + \cdots + \left[ A_m \frac{m\pi}{L} \sinh \left( \frac{m\pi H}{L} \right) \right] \frac{L}{2} + 0 + 0 + \cdots \]

And, solving for \( A_m \):

\[ A_m = \frac{2}{m\pi \sinh \left( \frac{m\pi H}{L} \right)} \int_0^L f(x) \cos \left( \frac{m\pi x}{L} \right) \, dx \]

As for the textbook note about \( \int f(x) \, dx \), refer back to Equation 2, and integrate both sides to see that

\[ \int_0^L f(x) \, dx = \sum_{n=1}^{\infty} \left[ A_n \frac{n\pi}{L} \sinh \left( \frac{n\pi H}{L} \right) \right] \int_0^L \cos \left( \frac{n\pi x}{L} \right) \, dx = 0 \]

(c) So how do we find the constant? As hinted at, the solution to Laplace’s equation is the steady state solution to the heat equation: \( u_t = k \nabla^2 u \). Furthermore, given the boundary conditions and the condition on \( f(x) \), the overall energy in the rectangle is constant. Therefore, if the rectangle is denoted as region \( R \), then:

\[ \iint_R u(t, x, y) \, dA = \iint_R u(0, x, y) \, dA = \iint_R g(x, y) \, dA \]

Furthermore, this is the same as the integral of our sum over the same rectangle:

\[ \iint_R u(x, y) \, dA = A_0 \sum_{n=1}^{\infty} A_n \int_0^H \cosh \left( \frac{n\pi y}{L} \right) \, dy \int_0^L \cos \left( \frac{n\pi x}{L} \right) \, dx = A_0 LH \]

Therefore,

\[ A_0 = \frac{1}{LH} \iint_R g(x, y) \, dA \]

which may be what we expected?
2.5.3 First, we’ll consider solving Laplace’s equation in polar form, and see what we need to make this a valid solution outside a disk of radius $r$. Proceeding as usual, let $u(r, \theta) = R(r)T(\theta)$, and substitute this into the polar form of Laplace’s Equation. You can start with the text version or a more expanded version. You should get:

$$\frac{r^2 R'' + r R'}{R} = -\frac{T''}{T}$$

So we’ll go after $T$ first, and make the constant $\lambda$:

$$r^2 R'' + r R' - \lambda R = 0 \quad T'' + \lambda T = 0$$

The periodic boundary conditions translate to: $T(-\pi) = T(\pi)$ and $T'(-\pi) = T'(\pi)$. Using these, we look at our three cases for $\lambda$:

- If $\lambda = 0$, we have the following ODEs:
  
  $$r^2 R'' + r R' = 0 \quad \text{and} \quad T'' = 0$$

  For the DE in $R$, this is linear (or separable), homogeneous and first order (let $w = R'$).

  $$R'' = -\frac{R'}{r} \Rightarrow w' = \frac{w}{r} \Rightarrow \frac{1}{w} \, dw = -\frac{1}{r} \, dr \Rightarrow \ln(w) = -\ln(r) + C \Rightarrow w = \frac{A_0}{r}$$

  Therefore, $R' = A_0/r$, and

  $$R(r) = A_0 \ln(r) + A_1 \quad T = C_1$$

  For our solutions to remain finite as $|r| \to \infty$, then $A_0 = 0$, so this reduces to a constant solution.

- If $\lambda < 0$, then a little work will show that $T$ is still trivial.

- If $\lambda > 0$, then we have the following forms for $T, T'$:

  $$T(t) = C_1 \cos(\sqrt{\lambda} \theta) + C_2 \sin(\sqrt{\lambda} \theta)$$

  $$T'(t) = -\sqrt{\lambda} C_1 \sin(\sqrt{\lambda} \theta) + \sqrt{\lambda} C_2 \cos(\sqrt{\lambda} \theta)$$

  The first condition, $T(-\pi) = T(\pi)$ implies that

  $$C_2 \sin(\sqrt{\lambda} \pi) = 0$$

  The second condition, $T'(-\pi) = T'(\pi)$, implies that

  $$C_1 \sin(\sqrt{\lambda} \pi) = 0$$

  Therefore, we keep both sets of constants when $\lambda = n^2$ for $n = 1, 2, \cdots$, and

  $$T_n = A_n \cos(n \theta) + B_n \sin(n \theta)$$
Furthermore, solving $r^2 R'' + r R' - n^2 R = 0$ for $r$ gives us two distinct, real solutions: $r = \pm n$. Therefore,

$$R(r) = C_1 r^n + C_2 r^{-n}$$

Again, for the solution to be bounded as $r$ increases, we must have only the second part; therefore

$$R_n = r^{-n}$$

And our general solution is:

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} A_n r^{-n} \cos(n \theta) + B_n r^{-n} \sin(n \theta)$$

Putting in the last part, we find the coefficients using the orthogonality of the cosines on $[-\pi, \pi]$. That is,

$$u(a, \theta) = \ln(2) + 4 \cos(3 \theta) = A_0 + \sum_{n=1}^{\infty} A_n a^{-n} \cos(n \theta) + B_n a^{-n} \sin(n \theta)$$

Therefore, $A_0 = \ln(2)$, $A_3 = 4a^3$, and all the other coefficients are zero.

For part (b), see the text. The only thing that might be tricky is to recall that the interval is on the symmetric interval $[-\pi, \pi]$, and not on the “half interval” $[0, \pi]$.

2.5.5(a) We can start with the separation:

$$\frac{r^2 R'' + r R'}{R} = \frac{T''}{T} = \lambda$$

Where $0 \leq r \leq 1$ and $0 \leq \theta \leq \pi/2$, and

$$T'(0) = 0 \quad T(\pi/2) = 0 \quad R(1) \neq 0$$

We’ll also assume that the solutions are bounded at the origin.

Solving for $T$ for first:

$$T'' + \lambda T = 0$$

Go through the usual cases:

- Can $\lambda = 0$? If so, then $T$ must be trivial (easy to check).
- Can $\lambda < 0$? If so, then again $T$ must be trivial (be able to show this).
With $\lambda > 0$, we have, for $T$ and $T'$:

\[ T(\theta) = C_1 \cos(\sqrt{\lambda} \theta) + C_2 \sin(\sqrt{\lambda} \theta) \]

\[ T'(\theta) = -\sqrt{\lambda} C_1 \sin(\sqrt{\lambda} \theta) + \sqrt{\lambda} C_2 \cos(\sqrt{\lambda} \theta) \]

Applying the boundary conditions, $T'(0) = 0$ implies that $C_2 = 0$. The other condition, $T(\pi/2) = 0$, is true if:

\[ C_1 \cos \left( \sqrt{\lambda} \frac{\pi}{2} \right) = 0 \quad \Rightarrow \quad \sqrt{\lambda} \frac{\pi}{2} = \frac{(2k-1)\pi}{2}, \quad k = 1, 2, 3, \ldots \]

Or, the eigenvalues are $(2n-1)^2$ with eigenfunctions $\cos((2n-1)\theta)$:

\[ T_n = A_n \cos((2n-1)\theta) \]

With this value of $\lambda$, we get $\pm (2n-1)$ for the roots to the characteristic equation for the ODE in $R$. Therefore,

\[ R(r) = C_1 r^{(2n-1)} + C_2 r^{-(2n-1)} \text{ for } n = 1, 2, 3, \ldots \]

To keep the solutions bounded, we take $C_2 = 0$ so that the general solution is:

\[ u(r, \theta) = \sum_{n=1}^{\infty} A_n r^{2n-1} \cos((2n-1)\theta) \]

With $u(1, \theta) = f(\theta)$, we can determine the coefficients (it’s the usual integral for $[0, L]$, with $L = \pi/2$):

\[ A_n = \frac{4}{\pi} \int_0^{\pi/2} f(\theta) \cos(2(n-1)\theta) \, d\theta \]

2.5.7(a) We want to solve Laplace’s Equation on a circular wedge:

\[ 0 \leq r \leq a \quad 0 \leq \theta \leq \frac{\pi}{3} \]

In Part (a), we’ll solve using the conditions:

\[ u(r, 0) = 0 \quad u(r, \frac{\pi}{3}) = 0 \quad u(a, \theta) = f(\theta) \]

Now let $u(r, \theta) = R(r)\Theta(\theta)$, and substitute this into the polar form of Laplace’s Equation:

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \frac{\partial^2 u}{\partial \theta^2}} = 0 \]

which gives us:

\[ \frac{1}{r} \frac{\partial}{\partial r} (r\Theta R') + \frac{1}{r^2} \Theta'' R = 0 \quad \Rightarrow \quad r \frac{\partial}{\partial r} (r\Theta R') + \Theta'' R = 0 \quad \Rightarrow \quad r \frac{\partial}{R \frac{\partial}{\partial r}} (rR') = -\frac{\Theta''}{\Theta} \]
Anticipating a periodic function in \( \theta \), we’ll use a positive \( \lambda \): 
\[
\frac{r}{R} \frac{\partial}{\partial r} (rR') = \lambda \quad \Rightarrow \quad \frac{\Theta''}{\Theta} = \lambda 
\]
We’ll attack \( \Theta \) first (it has a full set of boundary conditions):
\[
\Theta'' + \lambda \Theta = 0 \quad \Rightarrow \quad r = \pm \sqrt{-\lambda}
\]
Go through the three cases!

- \( \lambda = 0 \). Then \( \Theta = C_1 \theta + C_2 \). With the conditions that \( \Theta(0) = \Theta(\pi/3) = 0 \), we get \( C_1 = C_2 = 0 \). No joy here.
- \( \lambda < 0 \). Then \( \Theta = C_1 \cosh(\theta) + C_2 \sinh(\theta) \). Like the last case, the conditions will force both \( C_1 = C_2 = 0 \) (check it out).
- \( \lambda > 0 \) is the case we’re after. In this case,
\[
\Theta(\theta) = A \cos(\sqrt{\lambda} \theta) + B \sin(\sqrt{\lambda} \theta)
\]
With \( \Theta(0) = 0 \), then \( A = 0 \). With the second condition, we get eigenvalues:
\[
\Theta(\pi/3) = 0 \quad \Rightarrow \quad B \sin\left(\frac{\pi \sqrt{\lambda}}{3}\right) = 0 \quad \Rightarrow \quad \frac{\pi \sqrt{\lambda}}{3} = n\pi \text{ for } n = 1, 2, 3, \cdots
\]
Therefore, \( \lambda = 9n^2 \) for \( n = 1, 2, 3, \cdots \).

So far, we have \( \Theta(\theta) = B_n \sin(3n\theta) \). Now we go after the function \( R \). Expanding where we left off,
\[
\frac{r}{R} \frac{\partial}{\partial r} (rR') = \lambda \quad \Rightarrow \quad r^2 R'' + rR' - 9n^2 R = 0
\]
With the ansatz that \( R(r) = r^k \), we get the characteristic equation:
\[
k(k - 1) + k - 9n^2 = 0 \quad \Rightarrow \quad k^2 = 9n^2 \quad \Rightarrow \quad k = \pm 3n
\]
Therefore,
\[
R(r) = C_n r^{3n} + D_n r^{-3n}
\]
We want our solution to be bounded at the origin, \(|u(0, \theta)| < \infty\), so therefore, we take only the positive root. We’ll also collate the constants in our solution, so that
\[
u(r, \theta) = \sum_{n=1}^{\infty} B_n r^{3n} \sin(3n\theta)
\]
For the coefficients, we can derive the formulas needed by recalling the orthogonality of the sines on the interval \([0, L] = [0, \pi/3] \):
\[
B_n a^{3n} = \frac{2}{L} \int_0^L f(\theta) \sin(3n\theta) \, d\theta \quad \Rightarrow \quad B_n = \frac{6}{\pi a^{3n}} \int_0^{\pi/3} f(\theta) \sin(3n\theta) \, d\theta
\]
2.5.10 Consider Poisson’s equation \( \nabla^2 u = g(x) \) on some region \( R \) with \( u = f(x) \) on the boundary of \( R \). Suppose that \( u \) is one solution, and prove the solution is unique (using the Maximum Principle). NOTE: The variable \( x \) here is possibly a vector, but \( u, g, f \) are all scalar functions.

SOLUTION: Suppose that \( w \) is another solution. Then our claim is that the function \( u - w \) satisfies Laplace’s equation:

\[
\nabla^2 (u - w) = \nabla^2 u - \nabla^2 w = g(x) - g(x) = 0
\]

Furthermore, on the boundary, \( u - w \) is zero, since on the boundary:

\[
u - w = f(x) - f(x) = 0
\]

Now, the maximum principle states that, in steady state (which \( u - w \) is, since it solves Laplace’s equation), the maximum and minimum occur on the boundary. Therefore, the maximum (and minimum) difference between \( u \) and \( w \) is 0. Therefore,

\[
u = w
\]

and the solution to Poisson’s equation is unique.