Homework: Sections 7.8-7.9

The homework assigned was the “Extra Practice” with Bessel functions (first problem was non-Maple, the rest was Maple). The first solution is in the solution for Homework Set 10.
For Section 7.8, we had 1, 2, 5(d,f), 7. and for 7.9: 1(a,b), 2(a,b), 5*.

7.8.1 (I meant to assign (a-c) only)

The boundary value problem for a vibrating annular membrane $1 < r < 2$ is

$$\frac{d}{dr} \left( r \frac{df}{dr} \right) + \left( \lambda r^2 - \frac{m^2}{r^2} \right) f = 0$$

with $f(1) = f(2) = 0$, and $m = 0, 1, 2, \ldots$.

(a) Show that $\lambda > 0$.

SOLUTION: This is a Sturm-Liouville problem. Considering the Rayleigh quotient, any function that satisfies the boundary condition will be zero along the boundary so that, as is typical,

$$\lambda = \frac{\iint_R |\nabla \phi|^2 \, dxdy}{\iint_R \phi^2 \, dxdy} \geq 0$$

Can $\lambda = 0$? If so, then $\phi$ is constant, and since $\phi = 0$ on the boundary, then $\phi = 0$ everywhere.
Therefore, $\lambda > 0$.

(b) Obtain an expression that determines the eigenvalues.

SOLUTION: The solution to the BVP is

$$f(r) = C_1 J_m(\sqrt{\lambda}r) + C_2 Y_m(\sqrt{\lambda}r)$$

with $f(1) = f(2) = 0$, we have the system of equations:

$$0 = C_1 J_m(\sqrt{\lambda}) + C_2 Y_m(\sqrt{\lambda})$$
$$0 = C_1 J_m(2\sqrt{\lambda}) + C_2 Y_m(2\sqrt{\lambda})$$

For this system to have a non-trivial answer, the determinant of the coefficient matrix must be zero (so that the coefficient matrix is not invertible):

$$J_m(\sqrt{\lambda})Y_m(2\sqrt{\lambda}) - J_m(2\sqrt{\lambda})Y_m(\sqrt{\lambda}) = 0$$

It’s easy to plot this expression in Maple, for example, if we needed to estimate the zeros.
(c) For what value of \( m \) does the smallest eigenvalue occur? (Do this problem graphically) We see that for \( m = 0 \), the smallest eigenvalue \( \lambda \approx 1.77 \).

7.8.2 Consider the heat equation on a quarter circle

\[ u_t = k \nabla^2 u \]

**BCs**

\[ u(r, 0, t) = 0 \quad u(r, \pi/2, t) = 0 \]

**ICs**

\[ u(r, \theta, 0) = G(r, \theta) \]

(a) Find the BVP in \( r \).

**SOLUTION:** Using separation of variables, define \( u(r, \theta, t) = f(r)g(\theta)h(t) \). Substituting this into the polar form of the Laplacian will give us:

\[
fg'k = k \left( \frac{1}{r} \frac{\partial}{\partial r} (rf'gh) + \frac{1}{r^2} (fg''h) \right)
\]

Divide both sides by \( kfg \), and we get:

\[
\frac{h'}{kh} = \frac{1}{rf} (rf')' + \frac{1}{r^2} \frac{g''}{g} = -\lambda
\]

Therefore, multiplying the second part of the equation by \( r \),

\[
h' = -k\lambda h \quad \frac{r}{f} (rf')' + \frac{g''}{g} = -\lambda r^2
\]

so that

\[
\frac{g''}{g} = -\lambda r^2 - \frac{r}{f} (rf')' = -\mu
\]

That gives us the ODE in \( \theta \): \( g'' = -\mu g \), and for the radial equation we have:

\[ -(\lambda r^2 - \mu) = \frac{r}{f} (rf')' \]

which is equivalent to the form in the text. From the given boundary conditions, we have \( f(a) = 0 \) and \( f(0) \) is bounded.

(b) The radial BVP is a Bessel equation, if we take \( \mu = m^2 \). In that case, the solution to the BVP is

\[ f(r) = C_1 J_m(\sqrt{\lambda}r) \]

and with \( f(a) = 0 \), we have

\[ \sqrt{\lambda}a = z_{mn} \quad \Rightarrow \quad \lambda = \left( \frac{z_{mn}}{a} \right)^2 \]
(c) Part (c) notes that, since \( z_{m0} \) is the first zero of the Bessel function of order \( m \), then
\[
J_m \left( \frac{z_{m0}}{a} r \right)
\]
will not be zero between \( r = 0 \) and \( r = a \).

(d) Solve the IVP.
SOLUTION: We’ve already solved the radial equation:
\[
f_{mn}(r) = J_m \left( \sqrt{\lambda_{mn}} r \right)
\]
Now, the general solution and BCs for \( g \) are:
\[
g(\theta) = C_1 \cos(m\theta) + C_2 \sin(m\theta) \quad g(0) = 0, \quad g(\pi/2) = 0
\]
Putting the BCs in, we get \( C_1 = 0 \) and \( m = 2k \) (\( m \) must be an even integer- Since we’re already using \( k \), we’ll substitute \( 2m \) for \( m \)). so we could write:
\[
g_m(\theta) = \sin(2m\theta)
\]
and finally, in time we have
\[
T_{mn}(t) = e^{-k\lambda_{mn} t}
\]
Therefore, the overall solution is the superposition (we won’t use \( m = 0 \) since that will simply zero out the term in the sum):
\[
u(r, \theta, t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} A_{mn} J_{2m} \left( \sqrt{\lambda_{mn}} r \right) \sin(2m\theta) e^{-k\lambda_{mn} t}
\]
With \( u(r, \theta, 0) = G(r, \theta) \), we have (swapping the sum for convenience):
\[
G(r, \theta) = \sum_{m=1}^{\infty} \left[ \sum_{n=0}^{\infty} A_{mn} J_{2m} \left( \sqrt{\lambda_{mn}} r \right) \right] \sin(2m\theta) = \sum_{m=1}^{\infty} F_m(r) \sin(2m\theta)
\]
so that
\[
F_m(r) = \frac{\int_{0}^{\pi/2} G(r, \theta) \sin(2m\theta) d\theta}{\int_{0}^{\pi/2} \sin^2(2m\theta) d\theta} = \frac{4}{\pi} \int_{0}^{\pi/2} G(r, \theta) \sin(2m\theta) d\theta
\]
And
\[
A_{mn} = \frac{\int_{0}^{a} F_m(r) J_{2m} \left( \sqrt{\lambda_{mn}} r \right) r \, dr}{\int_{0}^{a} J_{2m}^2 \left( \sqrt{\lambda_{mn}} r \right) r \, dr}
\]
7.8.5(d,f) Use Maple to sketch these.

7.8.7 This is a change of variables problem (good practice!).
Given Bessel’s equation
\[ z^2 f''(z) + z f'(z) + (z^2 - m^2) f(z) = 0 \]
Let \( f(z) = \frac{y(z)}{\sqrt{z}} = y z^{-1/2} \)
and find the corresponding ODE in terms of \( y \).
SOLUTION: This is a straightforward application of the product/chain rules.
\[
    f'(z) = y' z^{-1/2} - \frac{1}{2} y z^{-3/2}
\]
so \( z f'(z) = y' \sqrt{z} - \frac{1}{2} y z^{-1/2} \). Similarly,
\[
    f''(z) = y'' z^{-1/2} - y' z^{-3/2} + \frac{3}{4} y z^{-5/2}
\]
so that \( z^2 f''(z) = y'' z^{3/2} - y' z^{1/2} + \frac{3}{4} y z^{-1/2} \). Put these together, divide by \( z^{3/2} \) and you’ll get the desired result (just a little algebra).

7.9.1 Before we look at 7.9.1/7.9.2, we note that the separation of variables stage will be the same for this problem and the next. We summarize that here:
In each problem, let \( u(r, \theta, z) = f(r) g(\theta) h(z) \). Substituting this into Laplace’s equation in polar coordinates results in the following ODEs:
\[
\bullet \text{ Height } z: \quad h'' = \lambda h
\]
\[
\bullet \text{ Angle } \theta: \quad g'' = -m^2 g
\]
The text gives physical reasons for \( g \), but we could solve for it as well given boundary conditions. In particular,
\[
g_m(\theta) = C_1 \cos(m \theta) + C_2 \sin(m \theta)
\]
\[
\bullet \text{ Radius } r: \quad r (r f')' + (\lambda r^2 - m^2) f = 0
\]
Furthermore, here we assume bounded solutions, so \( f_{mn} = J(\sqrt{\lambda_{mn}} r) \)
Now we need the individual boundary conditions to finish our solutions.
(a) \( u(r, \theta, 0) = \alpha(r, \theta) \quad u(r, \theta, H) = 0 \quad u(a, \theta, z) = 0 \)
This means that we have our usual conditions on $\lambda$:

$$J(\sqrt{\lambda_{mn}}a) = 0 \quad \Rightarrow \quad \lambda_{mn} = \frac{z_{mn}^2}{a^2}$$

and therefore, the general solution in $h$ uses hyperbolic sine and cosine. Because we want the expression to be 0 at $z = H$, we’ll use the $H - z$ argument. That is,

$$g(z) = C_1 \cosh(\sqrt{\lambda_{mn}}(H - z)) + C_2 \sinh(\sqrt{\lambda_{mn}}(H - z))$$

with $g(H) = 0$, $C_1 = 0$, so the eigenfunction is the hyperbolic sine.

We don’t have any extra conditions on $\theta$, so we’ll need to keep both of those. Therefore, we have:

$$u(r, \theta, z) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (A_{mn} \cos(m\theta) + B_{mn} \sin(m\theta)) J_m(\sqrt{\lambda_{mn}}r) \sinh(\sqrt{\lambda_{mn}}(H - z))$$

For the initial conditions, we could write this as:

$$\alpha(r, \theta) = \sum_{m=0}^{\infty} F_m(r) \cos(m\theta) + G_m(r) \sin(m\theta)$$

so that

$$F_0(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \alpha(r, \theta) \, d\theta$$

$$F_m(r) = \frac{1}{\pi} \int_{-\pi}^{\pi} \alpha(r, \theta) \cos(m\theta) \, d\theta, \quad m \neq 0$$

and

$$G_m(r) = \frac{1}{\pi} \int_{-\pi}^{\pi} \alpha(r, \theta) \sin(m\theta) \, d\theta$$

Once these are computed, we can compute the constants $A_{mn}, B_{mn}$:

$$F_m(r) = \sum_{n=1}^{\infty} A_{mn} J_m(\sqrt{\lambda_{mn}}r) \sinh(\sqrt{\lambda_{mn}}(H))$$

so that

$$A_{mn} = \frac{1}{\sinh(\sqrt{\lambda_{mn}}(H))} \frac{\int_0^{a} F_m(r) J_m(\sqrt{\lambda_{mn}}r) \, r \, dr}{\int_0^{a} J_m^2(\sqrt{\lambda_{mn}}r) \, r \, dr}$$

and similarly,

$$B_{mn} = \frac{1}{\sinh(\sqrt{\lambda_{mn}}(H))} \frac{\int_0^{a} G_m(r) J_m(\sqrt{\lambda_{mn}}r) \, r \, dr}{\int_0^{a} J_m^2(\sqrt{\lambda_{mn}}r) \, r \, dr}$$
(b) Is part (b) identical to (a)? Almost- The only change is that
\[ \alpha(r, \theta) = \alpha(r) \sin(7\theta) \]

What does this change to the previous solution? You should focus on the computation of \( F_m(r) \) and \( G_m(r) \). Since we are multiplying our basis functions by \( \sin(7\theta) \), these coefficients are all zero except for \( G_7(\theta) \). Therefore, our solution is much simpler- We replace the sum through \( m \) by \( m = 7 \):

\[ u(r, \theta, z) = \sum_{n=1}^{\infty} B_n \sin(7\theta) J_7(\sqrt{\lambda_7 n} r) \sinh(\sqrt{\lambda_7 n}(H - z)) \]

where \( B_n = B_{7n} \) as in the previous answer.

7.9.2 Very similar to 7.9.1- If you have questions, see me.

7.9.5* Included below (also in HW set 10)
Determine the three ODEs obtained by separating variables for Laplace’s equation in spherical coordinates:

\[ 0 = \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{\sin(\phi)} \frac{\partial}{\partial \phi} \left( \sin(\phi) \frac{\partial u}{\partial \phi} \right) + \frac{1}{\sin^2(\phi)} \left( \frac{\partial^2 u}{\partial \theta^2} \right) \]

With \( u(r, \theta, \phi) = f(r)q(\theta)g(\phi) \), we substitute in:

\[ 0 = \frac{\partial}{\partial r} \left( r^2 f' q g \right) + \frac{1}{\sin(\phi)} \frac{\partial}{\partial \phi} \left( \sin(\phi) f q g' \right) + \frac{1}{\sin^2(\phi)} \left( f q'' g \right) \]

Following through with the usual separation, we should end up with something like:

\[ \frac{1}{f} \frac{\partial}{\partial r} \left( r^2 f' \right) = -\lambda \quad \Rightarrow \quad r^2 f'' + 2r f' + \lambda f = 0 \]

(which is a Cauchy-Euler equation) and

\[ \frac{1}{g \sin(\phi)} \frac{\partial}{\partial \phi} \left( \sin(\phi) q' \right) + \frac{1}{q \sin^2(\phi)} (q'') = -\lambda \]

To separate, we need to multiply by \( \sin^2(\phi) \) so that \( q(\theta) \) is by itself:

\[ \frac{\sin(\phi)}{g} \frac{\partial}{\partial \phi} \left( \sin(\phi) q' \right) + \frac{q''}{q} = -\lambda \sin^2(\phi) \]

And now the variables separate:

\[ \frac{\sin(\phi)}{g} \frac{\partial}{\partial \phi} \left( \sin(\phi) q' \right) + \lambda \sin^2(\phi) = -\frac{q''}{q} = \mu \]

(The sign of this is arbitrary at this point) Then:
• Radius: \( r^2 f'' + 2rf' + \lambda f = 0 \)
• Angle \( \theta \): \( q'' = -\mu q \)
• Angle \( \phi \):
\[
\sin(\phi) \frac{\partial}{\partial \phi} (\sin(\phi)g') + (\lambda \sin^2 \phi - \mu)g = 0
\]