FPI with Systems (Exercise Solutions)

1. Let \( x_{n+1} = Tx + c \). Show that

\[
x_n = T^n x_0 + (T^{n-1} + T^{n-2} + \cdots + T^2 + T + I) c
\]

**SOLUTION:** Go through the first few of them:

\[
x_1 = Tx_0 + c
\]
\[
x_2 = Tx_1 + c = T(Tx_0 + c) + c = T^2 x_0 + (T + I)c
\]
\[
x_3 = Tx_2 + c = T(T^2 x_0 + (T + I)c) + c = T^3 x_0 + (T^2 + T + I)c
\]

and we see the general pattern:

\[
x_n = T^n x_0 + (T^{n-1} + T^{n-2} + \cdots + T^2 + T + I) c
\]

2. Show that, if \( x_{n+1} = Tx_n + c \) and \( r = Tr + c \), then using an induced matrix norm,

\[
\|x_k - r\| \leq \|T\|^k \|x_0 - r\|
\]

**SOLUTION:** Easiest to show working up from \( k = 1 \):

\[
\|x_1 - r\| = \|Tx_0 + c - Tr - c\| = \|T(x_0 - r)\| \leq \|T\| \|x_0 - r\|
\]

Similarly,

\[
\|x_2 - r\| = \|Tx_1 + c - Tr - c\| = \|T(x_1 - r)\| \leq \|T\| \|x_1 - r\| \leq \|T\|^2 \|x_0 - r\|
\]

And one more:

\[
\|x_3 - r\| = \|T(x_2 - r)\| \leq \|T\|^3 \|x_0 - r\|
\]

Continuing in this fashion, we get the desired result:

\[
\|x_k - r\| \leq \|T\|^k \|x_0 - r\|
\]

3. Exercise 1(a), 1(c) (only Jacobi). Also see the code online that will split the matrix \( A \) for you.