**Homework: Vector Norms**

**How to Measure Distances Between Things in \( \mathbb{R}^n \)**

**Definition:** The norm of a vector \( x \in \mathbb{R}^n \) is a function from \( \mathbb{R}^n \) to \( \mathbb{R} \), denoted by \( \| \cdot \| \), so that the following are properties are satisfied:

1. \( \|x\| \geq 0 \), and \( \|x\| = 0 \) iff \( x = 0 \).
2. \( \|cx\| = |c|\|x\| \), for all scalars \( c \).
3. \( \|x + y\| \leq \|x\| + \|y\| \) (Triangle Inequality)

**Definition:** Given a vector space \( X \) and a norm \( \| \cdot \| \), the distance between two vectors \( x, y \) is \( \|x - y\| \).

**Definition:** The \( p \)-norm of a vector \( x \in \mathbb{R}^n \) is

\[
\|x\|_p = (|x_1|^p + |x_2|^p + \ldots + |x_n|^p)^{1/p}
\]

In particular,

\[
\|x\|_1 = |x_1| + |x_2| + \ldots + |x_n| \quad \|x\|_2 = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}
\]

And, we will define the “infinity norm” as:

\[
\|x\|_\infty = \max_i \{|x_i|\}
\]

**Exercise Set 1**

1. Show that \( \|x\|_\infty \) satisfies the three parts in the definition of a norm.

2. What if two vectors were the same except for a single coordinate (without loss of generality, make it the first coordinate). What would the distance between them be under the \( 1-, 2- \) and \( \infty \) norms?

3. If two vectors (in \( \mathbb{R}^n \)) are within \( \epsilon \) of each other in the \( \infty \) norm, how close together are they in the \( 1- \) norm? (Hint: Start with definitions)

4. If two vectors (in \( \mathbb{R}^n \)) were within \( \epsilon \) of each other in the \( 1- \) norm, how close together are they in the \( \infty \) norm?

5. Let \( F(c) = F(c_1,c_2,\ldots, c_n) = c_1a_1 + c_2a_2 + \ldots + c_na_n \), where every \( c_i \geq 0 \), \( \sum c_i = 1 \), and \( a_1, a_2, \ldots, a_n \) are given, fixed numbers. Find the maximum value of \( F \), and the \( c \) where it occurs. (If you get stuck, try putting in some numbers for the \( a_i \) and simplifying the problem- For example, find the max of \( 3c_1 + 5c_2 \))

6. Same function as before, except that the values of \( c_i \) are changed: This time, the only restriction is \( |c_i| = 1 \) for each \( i \).